

Parameterized Complexity of Conflict-free Graph Coloring

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Conflict-free graph coloring

q -Closed Neighborhood Conflict-Free Coloring

q -CNCF-Coloring

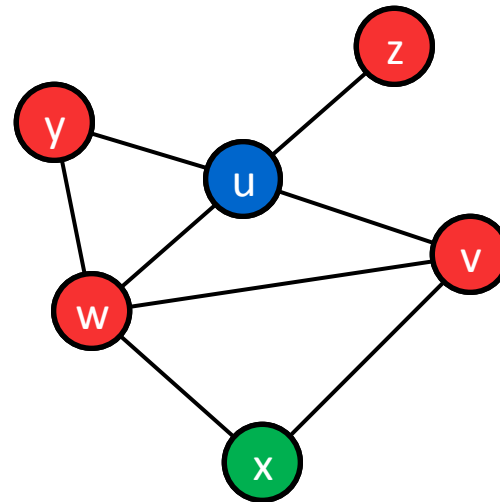
Input: A graph G (with vertex cover S)

Parameter: $k = |S|$

Question: Is it possible to assign every vertex in G a color from $\{1, \dots, q\}$, such that for all v , there is a color occurring exactly once in $N[v]$?

Related: q -ONCF-Coloring

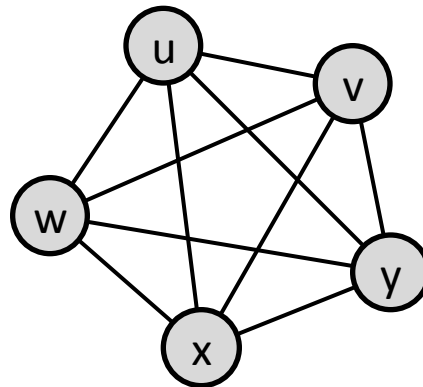
- Considers **open neighborhoods** instead



CNCF-Coloring

How many colors do we need to CNCF-color

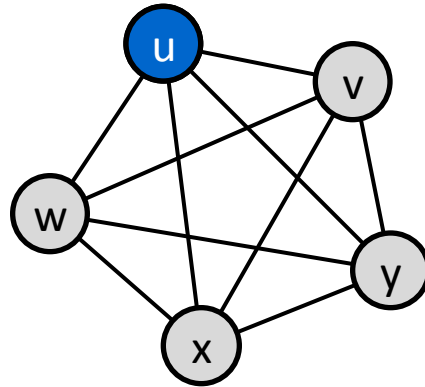
- A clique on k vertices?



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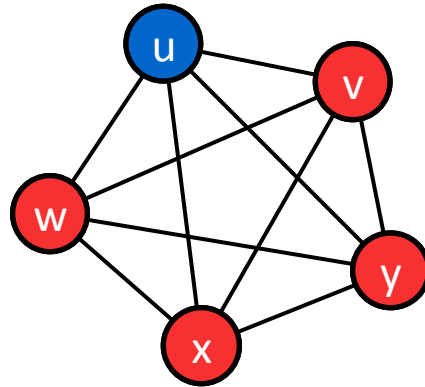
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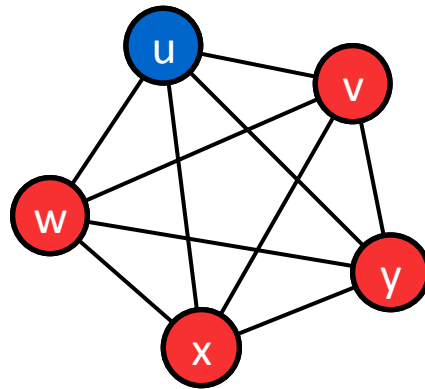
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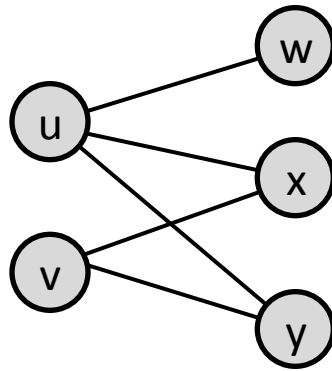
- A clique on k vertices? 2



CNCF-Coloring

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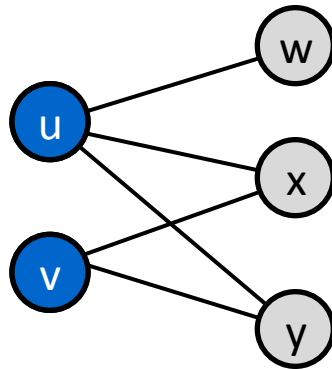
- A clique on k vertices? 2
- A bipartite graph?



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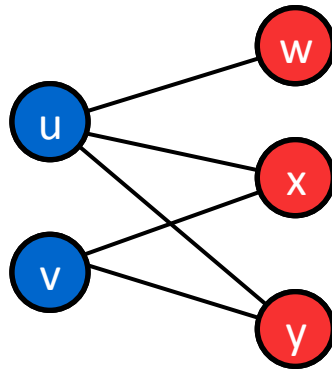
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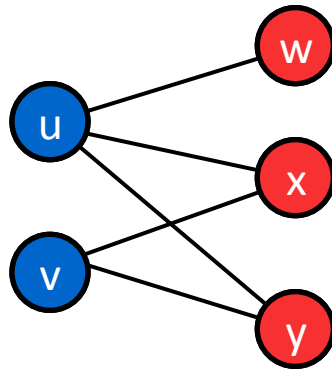
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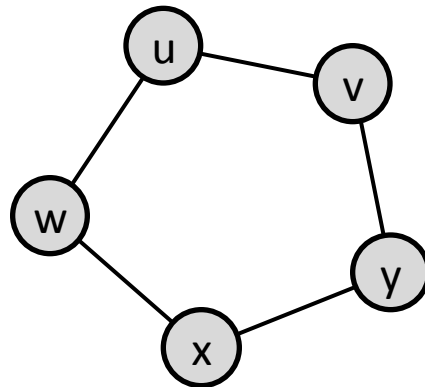
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CNCF-Coloring

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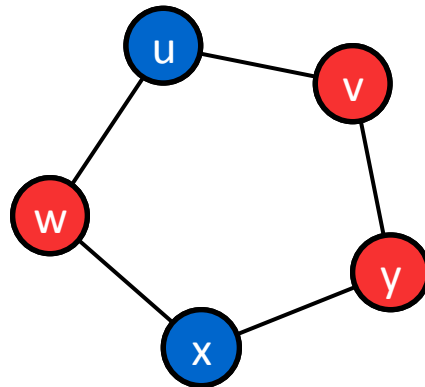
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- An odd cycle?



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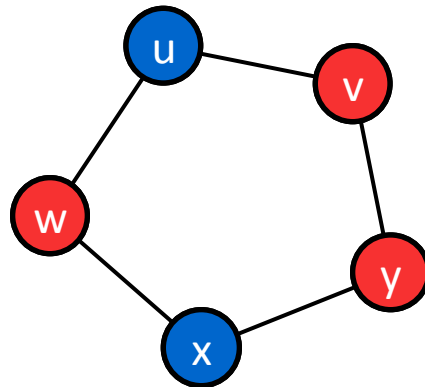
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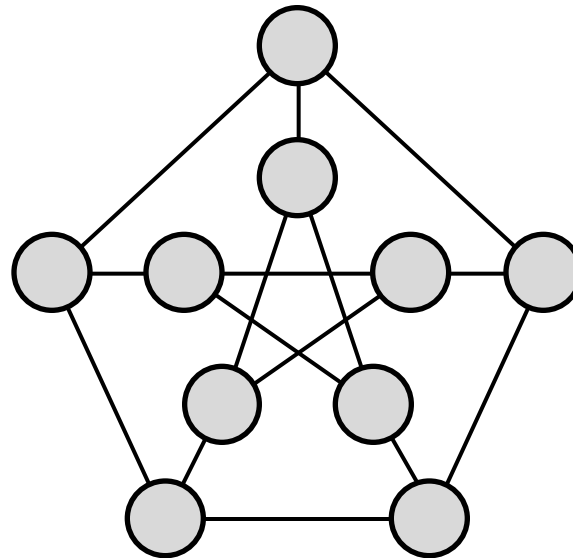
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CNCF-Coloring

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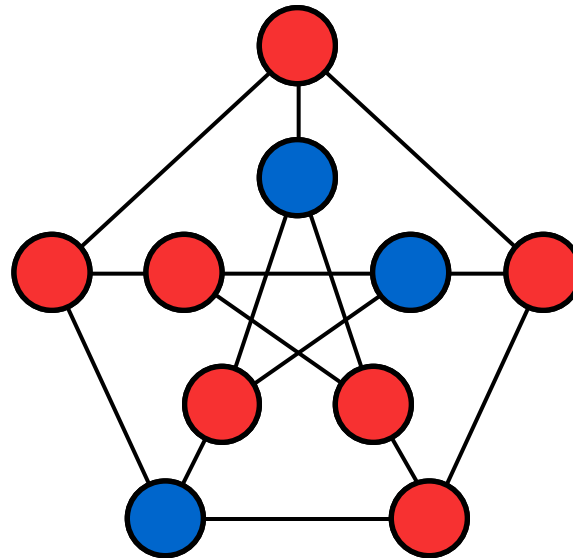
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- A bipartite graph? 2
- An odd cycle? 2
- Petersen graph



CNCF-Coloring

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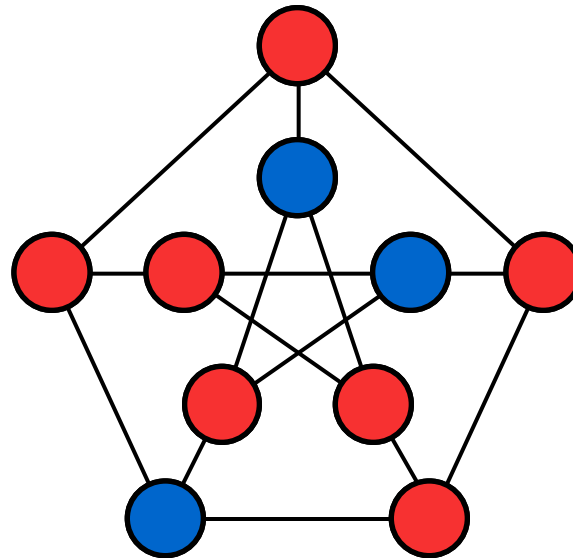
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CNCF-Coloring

How many colors do we need to CNCF-color

- A clique on k vertices? 2
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CNCF-Coloring

How many colors do we need to CNCF-color

- A clique on k vertices? 2
- A bipartite graph? 2
- An odd cycle? 2
- Petersen graph 2
- A k -colorable graph?

CNCF-Coloring

How many colors do we need to CNCF-color

- A clique on k vertices? 2
- A bipartite graph? 2
- An odd cycle? 2
- Petersen graph 2
- A k -colorable graph? k

Observation

Any proper coloring is a CNCF-Coloring

- Each vertex is the uniquely colored vertex in its closed neighborhood

CNCF-Coloring

How many colors do we need to CNCF-color

- A clique on k vertices? 2 $\leq k$
- A bipartite graph? 2 ≤ 2
- An odd cycle? 2 ≤ 3
- Petersen graph 2 ≤ 4
- A k -colorable graph? $k \leq k$

Observation

Any proper coloring is a CNCF-Coloring

- Each vertex is the uniquely colored vertex in its closed neighborhood

Background

- Special case of conflict-free coloring set systems
 - Sets given by closed (or open) neighborhoods
- Frequency assignment problems

Studied from a combinatorial perspective

- CNCF-coloring n -vertex graph takes $O(\log^2 n)$ colors
[Pach, Tardos 2009]
 - **Tight** [Glebov, Szabó, Tardos, 2014]

Background

NP-hard for $q \geq 2$ [Gargano and Rescigno, TCS, 2015]

- Study parameterized complexity and kernelizability
 - FPT parameterized by [Gargano and Rescigno, TCS, 2015]
 - Vertex Cover
 - Neighborhood diversity
 - Treewidth
 - In this talk: kernels!

Results

Kernelization parameterized by Vertex Cover

- 2-CNCF-Coloring has a polynomial kernel (up next)
- q -CNCF-Coloring has no polynomial kernel¹ for $q \geq 3$
- q -ONCF-Coloring has no polynomial kernel¹ for $q \geq 2$
 - Both results proven by cross-composition
 - See <https://arxiv.org/pdf/1905.00305>

¹Unless $NP \subseteq coNP/poly$

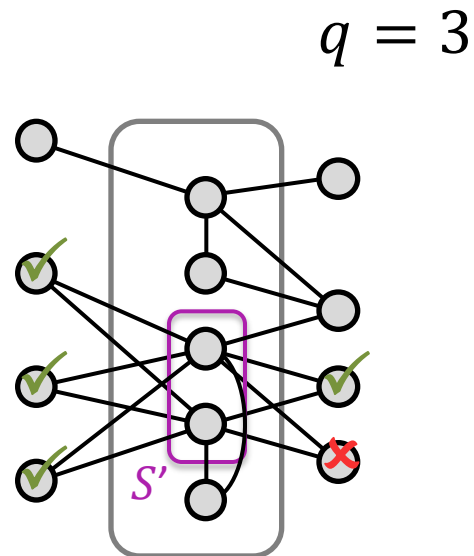
A general reduction rule

For q -CNCF-Coloring

[Based on Gargano and Rescigno, TCS 2015, Lemma 6]

Reducing number of twins

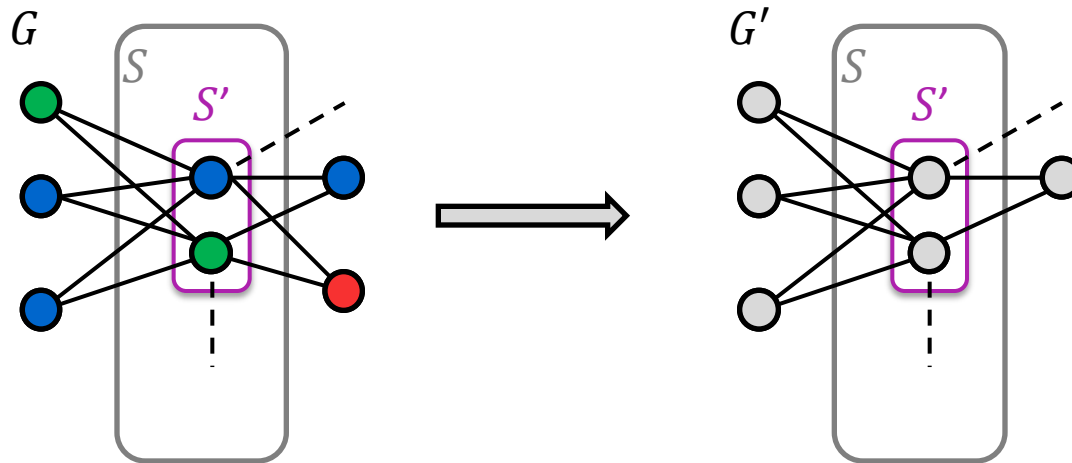
Let $S' \subseteq S$. Suppose there are $> q + 1$ vertices $v \notin S$ with $N(v) = S'$. Mark $q + 1$, remove the others.



Correctness

Let G' be the resulting graph

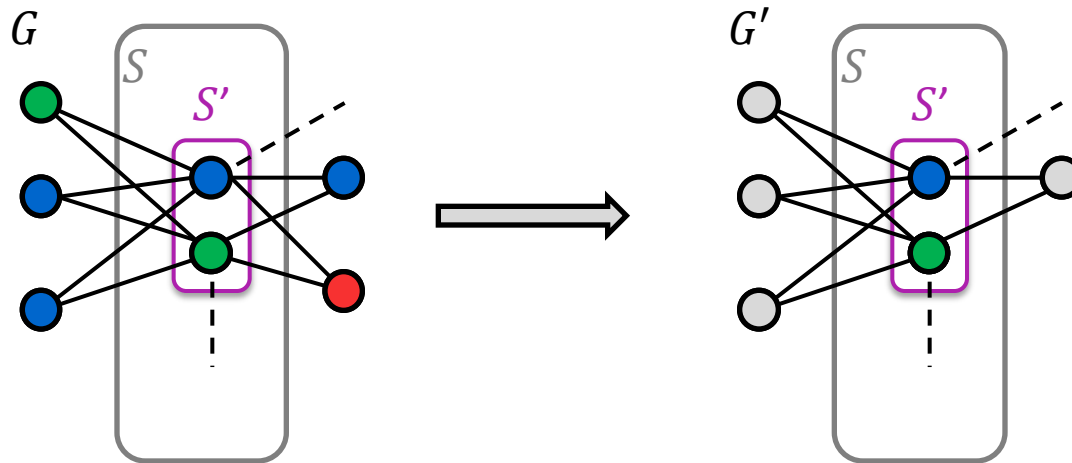
- Suppose G is q -CNCF-Colorable
 - Color G' similarly, ensuring that each vertex in S keeps its conflict-free neighbor (if any)



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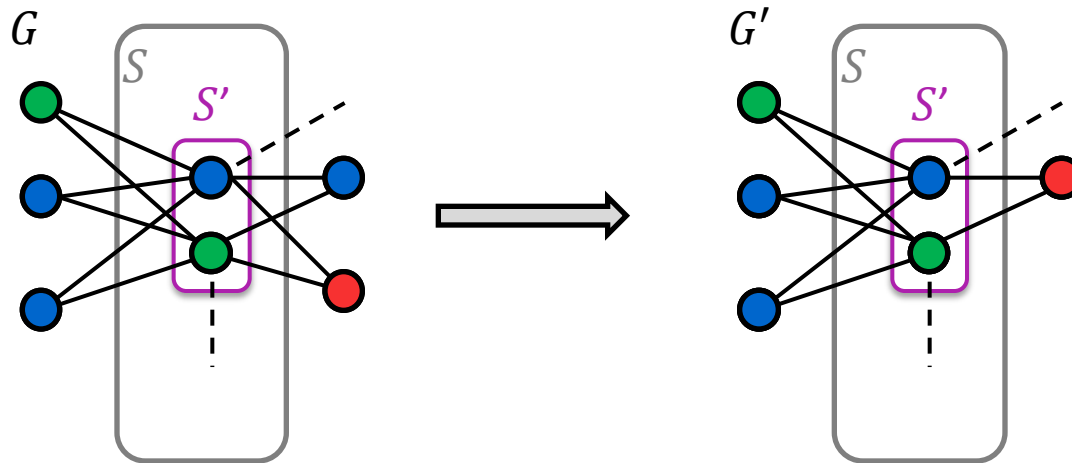
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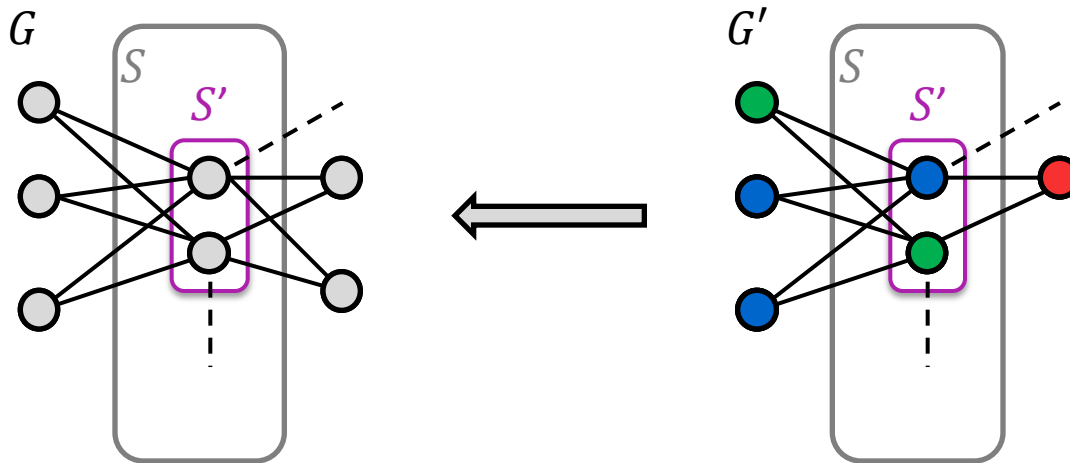
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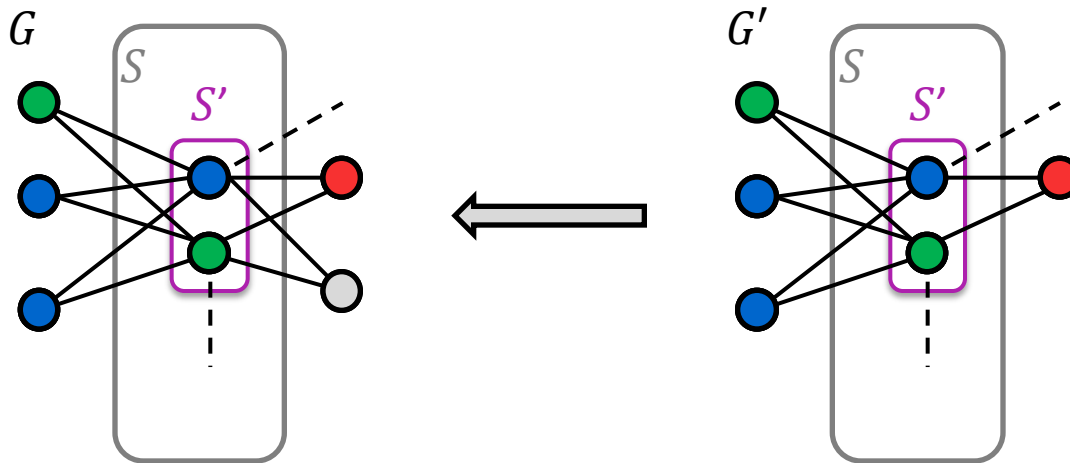
Correctness

- Suppose G' is q -CNCF-Colorable
 - Use the same coloring on G
 - Let v be a removed vertex, color v using a color already used twice in $N(v)$
 - Observe that such a color exists!



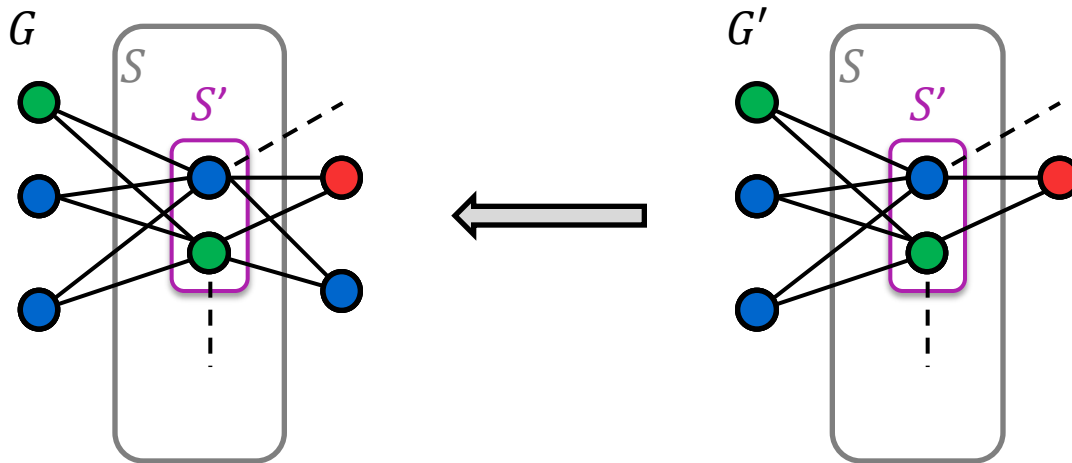
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Effectiveness

Suppose we apply this rule **exhaustively**, for all $S' \subseteq S$

Number of vertices in S

- k by definition

Number of vertices not in S

- Number of **degree- d vertices** not in S
 - At most $(q + 1) \binom{|S|}{d} = O(k^d)$
- Total $O(2^k)$

Exponential, unless we can somehow bound the number of **high-degree** vertices

Polynomial kernel for 2-CNCF-Coloring Extension

Parameterized by Vertex Cover size

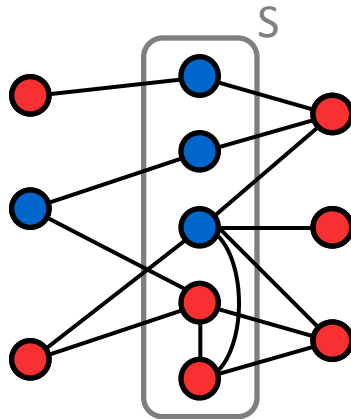
The extension problem

2-CNCF-Coloring Extension

Input: A graph G , with vertex cover S and partial coloring $c : S \rightarrow \{red, blue\}$

Parameter: $|S|$

Question: Can c be extended to a 2-CNCF-coloring of G ?

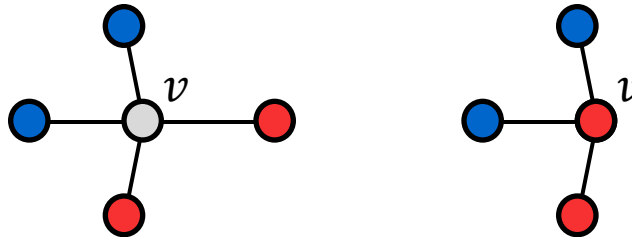


This is a **yes**-instance!

Trivial reduction rule

If at any point there is a vertex v whose neighborhood contains at least two red and two blue vertices under c

- Output **NO**



Removing low-degree vertices

Apply known reduction rule:

For every $S' \subseteq S$ of size at most 2

- Mark 3 vertices $u \notin S$ such that $N(u) = S'$
 - If there are less than three, mark all

Delete all unmarked vertices of degree 1 and 2

Number of vertices in S

- k (by definition)

Number of degree- ≤ 2 vertices not in S

- At most $3k + 3\binom{k}{2} = O(k^2)$

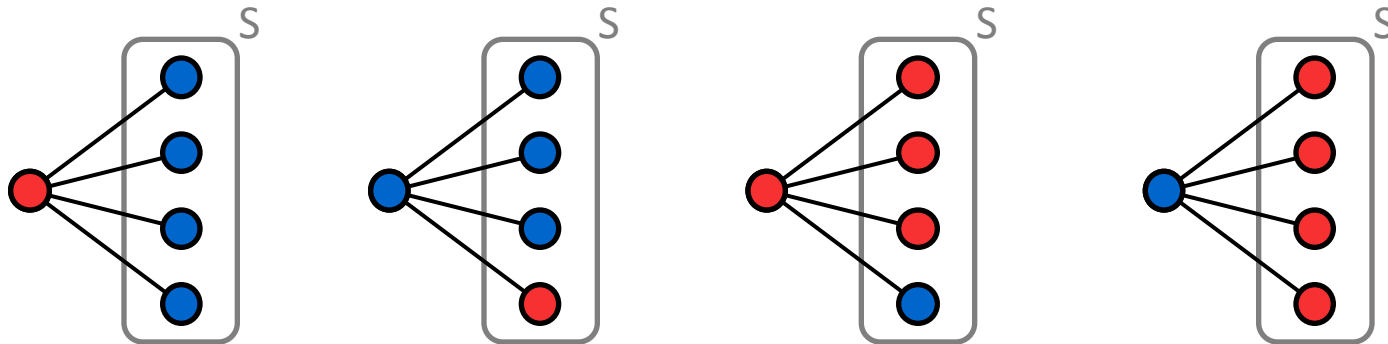
Number of degree- ≥ 3 vertices not in S

- Possibly many ☹️

Removing high-degree vertices

For all $v \notin S$ of degree at least 3 do the following.

- Extend c by coloring v such that $N[v]$ is conflict-free
 - This uniquely determines $c(v)$

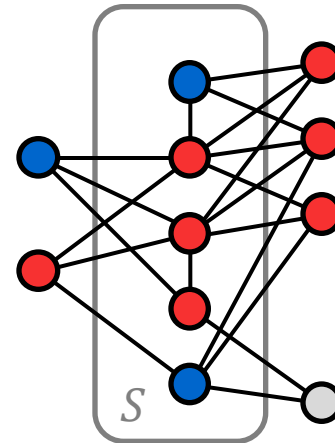


Marking procedure

For every $x \in S$, mark 2 red and 2 blue neighbors in $V(G) \setminus S$

- If there are no two red/blue neighbors, mark all red/blue neighbors

Delete all unmarked vertices
of degree ≥ 3

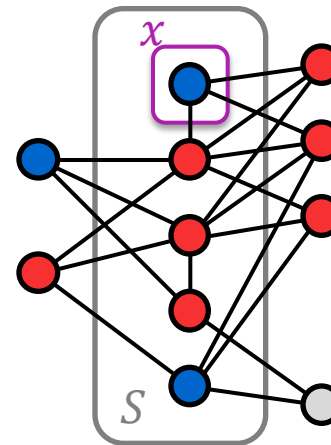


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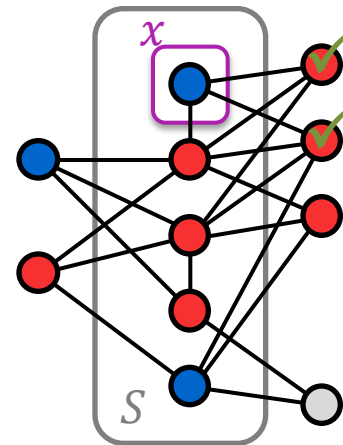


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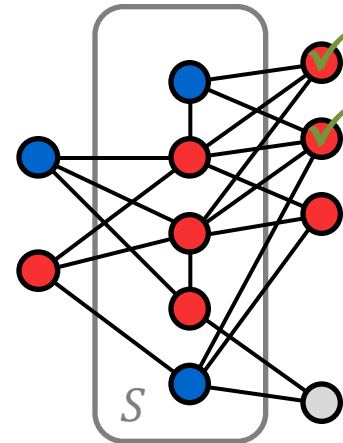


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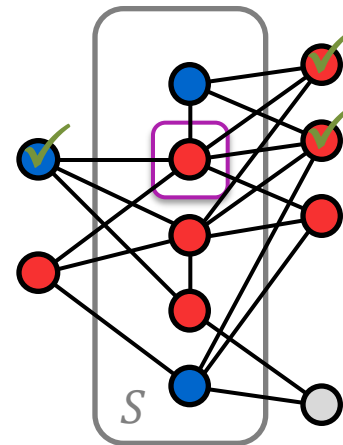


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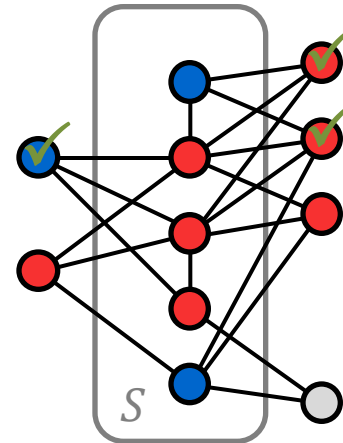


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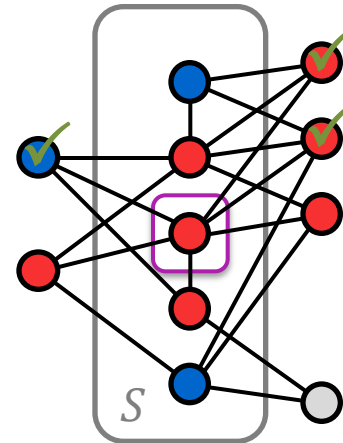


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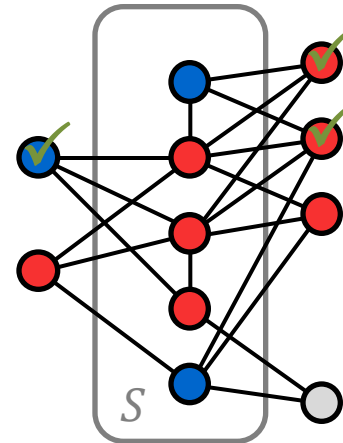


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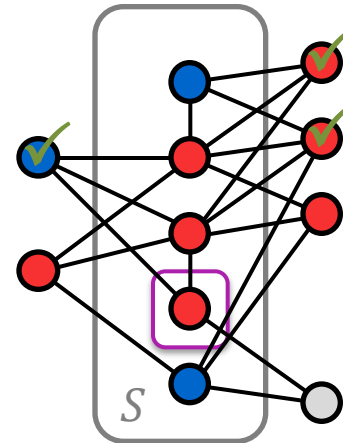


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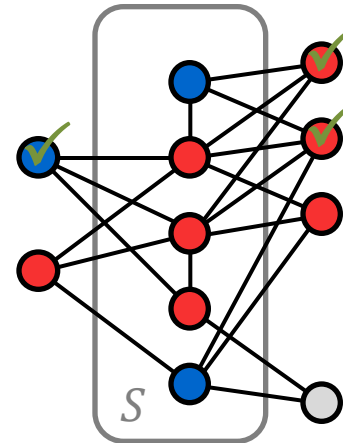


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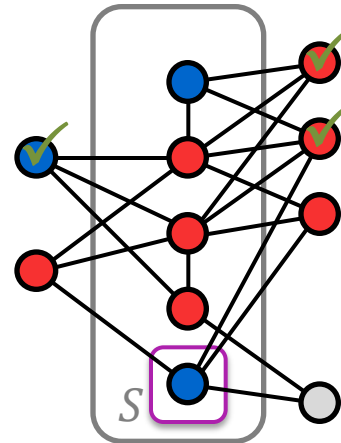


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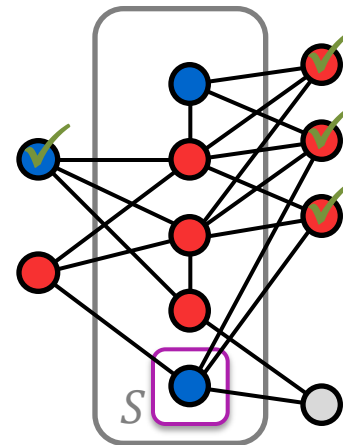


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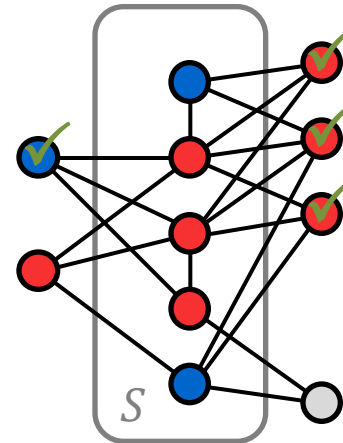


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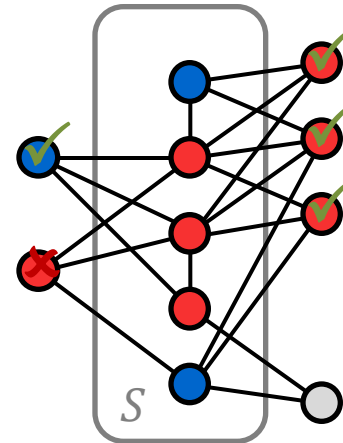


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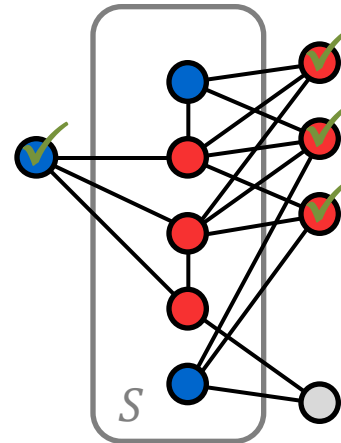


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Correctness

Suppose G' is CNCF-colorable

- Color G the same, color removed vertices as they were before removal
 - For $v \notin S$, this conflict-free colors $N[v]$ by definition
 - For $v \in S$, this never destroys the conflict-free coloring of $N[v]$

Suppose G is CNCF-colorable

- Use the same coloring for G'

Kernel size

We mark at most 4 vertices for each vertex in S

- At most $4k$ high-degree vertices not in S remain

Combined with removing low-degree vertices

- k vertices in S
- $O(k^2)$ degree- ≤ 2 vertices not in S
- $O(k)$ degree- ≥ 3 vertices not in S
- Total of $O(k^2)$ vertices

Theorem

2-CNCF-Coloring Extension parameterized by $|S|$ has a kernel of size $O(k^2 \log k)$, and no kernel¹ of size $O(k^{2-\varepsilon})$

¹ Unless $NP \subseteq coNP/poly$

Polynomial kernel for 2-CNCF-Coloring

Parameterized by Vertex Cover size

A generalized kernel

We give a generalized kernel to *d*-Polynomial root CSP

Input: A set L of equalities over variables X , where each equality is of the form $p(x_1, \dots, x_n) = 0$, where p is a polynomial of degree at most d

Parameter: The number of variables n

Question: Does there exist an assignment $\tau: X \rightarrow \{0,1\}$ satisfying all equalities in L ?

Example of 2-Poly root CSP:

$$\{x_1 + x_2 - 1 = 0, \quad x_1 * x_2 + x_2 * x_3 = 0\}$$

Satisfied by

$$\tau(x_1) = \tau(x_3) = 0, \tau(x_2) = 1$$

A generalized kernel

We give a generalized kernel to d -Polynomial root CSP

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Parameter: The number of variables n

Question: Does there exist an assignment $\tau: X \rightarrow \{0,1\}$ satisfying all equalities in L ?

Theorem [[more on this tomorrow](#), Jansen and Pieterse, MFCS 2016]

d -Polynomial root CSP has a kernel with $O(n^d)$ equalities, where n is the number of variables

- That is a subset of the original set of equalities!

Kernelization: General idea

Three steps

1. Reduce the number of low-degree vertices
2. Rewrite the problem to an instance of d -Poly root CSP
 - For some constant d
 - Using not too many variables
 - Tricky!
3. Apply (known) kernelization result for d -Poly root CSP
 - d -Polynomial root CSP has a kernel with $O(\#vars^d)$ equalities

Removing low-degree vertices

- Same as before: marking procedure to reduce the number of degree-1 and 2 vertices outside S
- Reduces their number to $O(k^2)$
- Add these to S
 - Technically, this increases $|S|$ to $O(k^2)$, but we ignore this for simplicity

Rewriting: Basics

Creating an instance of d -Poly root CSP (for some d)

- For each vertex v , create **variables** r_v and b_v
 - $r_v = 1$ means v is red, $b_v = 1$ means it is blue
 - Add the constraint that $r_v + b_v = 1$
- A **constraint** on the coloring of $N[v]$ for all v
 - Exactly one blue, or exactly one red vertex
 - Thus, $\sum_{u \in N[v]} r_u = 1$, or $\sum_{u \in N[v]} b_u = 1$
 - For all v add the constraint
 - $(1 - \sum_{u \in N[v]} r_u)(1 - \sum_{u \in N[v]} b_u) = 0$

Rewriting: Continued

So far, variables $\{r_v, b_v \mid v \in V(G)\}$, constraints

- For all v : $r_v + b_v = 1$
- For all v : $(1 - \sum_{u \in N[v]} r_u)(1 - \sum_{u \in N[v]} b_u) = 0$

Hereby

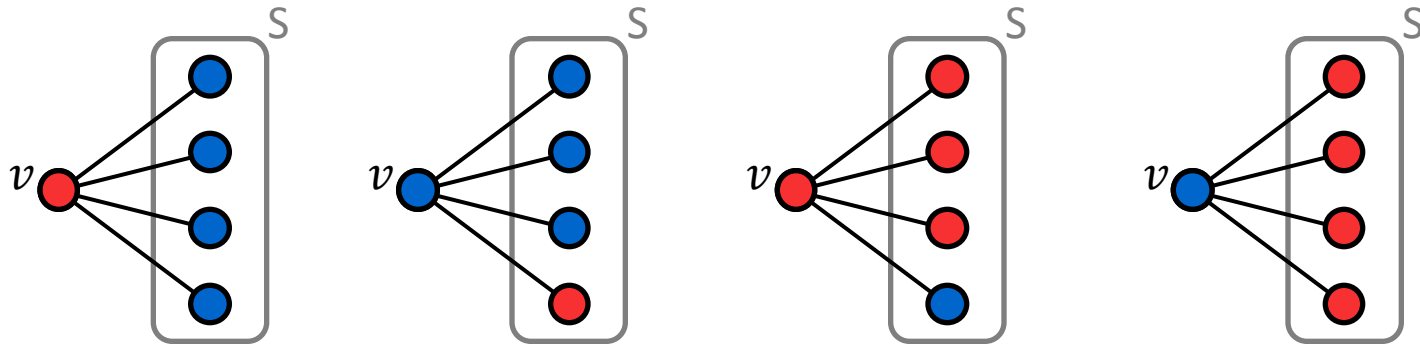
- The two problem instances are **equivalent**
- We use **low-degree** polynomials (degree-2)
- As **many variables** as vertices
 - Using the known kernel for d -Poly root CSP gives a kernel of size $O(n^2)$ (useless)

Plan: **reduce the number of variables** to $O(k)$?

Reducing the number of variables

Recall, each vertex $v \notin S$ has degree at least 3

- Its coloring is precisely determined by the colors of $N(v)$



Idea: write r_v as $f(r_{u_1}, \dots, r_{u_k}, b_{u_1}, \dots, b_{u_k})$ if $N(v) = \{u_1, \dots, u_k\}$

- For low-degree polynomial f
- Then $b_v = 1 - r_v = 1 - f(\dots)$
- Substituting r_v by $f(\dots)$, reduces the number of variables to $2|S| = 2k$

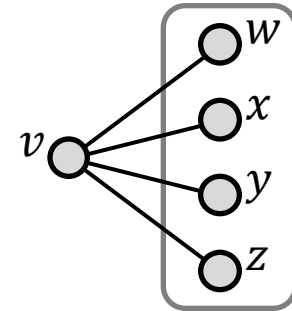
Reducing the number of variables

First, ensure that the neighborhood of $v \notin S$ is “ok”

- All blue, all red, one red, or one blue
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- $r_w + r_x + r_y + r_z = 3$ implies $r_v = 1$
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Let g such that $g(x) = 0$ if $x \in \{1,4\}$ and $g(x) = 1$ if $x \in \{0,3\}$

- Substitute $r_v = f(r_w, r_x, r_y, r_z) = g(r_w + r_x + r_y + r_z)$

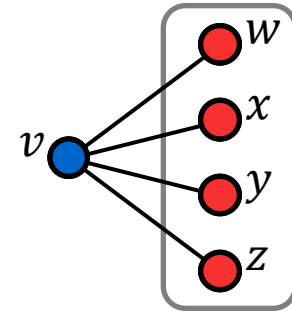
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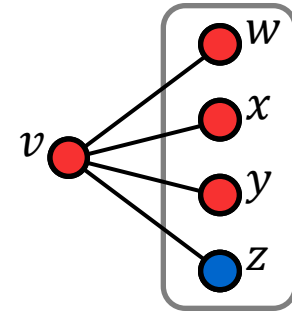
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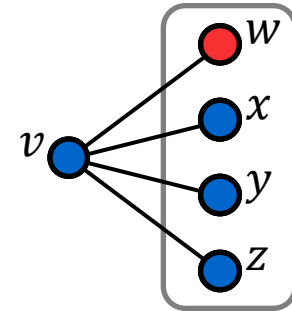
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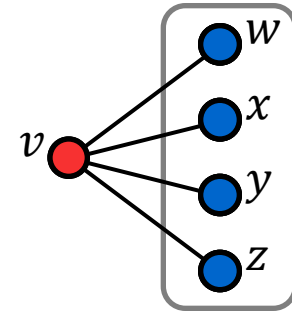
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Reducing the number of variables

In general, for $v \notin S$, find polynomial g s.t.

- $g(x) = 0$ if $x \in \{1, |N(v)|\}$
- $g(x) = 1$ if $x \in \{0, |N(v)| - 1\}$

Use interpolating polynomial for
 $\{1, 0\}, \{|N(v)|, 0\}, \{0, 1\}, \{|N(v)| - 1, 1\}$

- Has degree 3

Let $N(r_v) = \{u_1, \dots, u_m\}$

- Substitute r_v by $g(r_{u_1} + r_{u_2} + \dots + r_{u_m})$ in all equalities.

Size and correctness

Now: variables $\{r_v, b_v \mid v \in V(G)\}$, constraints

- For all v : $r_v + b_v = 1$
- For all v : $(1 - \sum_{u \in N[v]} r_u)(1 - \sum_{u \in N[v]} b_u) = 0$
 - With for $v \notin S$ r_v substituted by $g(\dots)$, b_v by $1 - g(\dots)$
- For all $v \notin S$:

$$(\sum_{u \in N(v)} r_u)(\sum_{u \in N(v)} b_u)(1 - \sum_{u \in N(v)} r_u)(1 - \sum_{u \in N(v)} b_u) = 0$$

Polynomials have degree ≤ 6

Replacing r_v by $g(\dots)$ is **safe**

- One direction, obvious
- Other direction: additional constraint ensures
 - $r_{u_1} + r_{u_2} + \dots + r_{u_m} \in \{0, 1, |N(v)| - 1, |N(v)|\}$
 - g chosen such that it $g(\dots) = r_v$

Kernelization

We obtained a d -Poly root CSP instance

- $d = 6$
- On $2k$ variables (actually, k variables suffices)

That is equivalent to the original instance

- Apply kernel for 6-Poly root CSP
 - Instance with $O(\#vars^d) = O(k^6)$ equalities
 - Can be encoded in $O(k^{10})$ bits

Theorem

2-CNCF-Coloring parameterized by Vertex Cover has a generalized kernel of size $O(k^{10})$

- Can be turned into normal kernel of polynomial size

Conclusion

- 2-CNCF-Coloring parameterized by vertex cover has a polynomial kernel
- q -CNCF-Coloring for $q \geq 3$ and q -ONCF-Coloring do not
 - Not even for the extension problem

Open questions

- Is the $O(k^{10})$ bound tight for 2-CNCF-Coloring?
 - Probably not
- Is there an “easier” kernel?

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THANK YOU