Elimination Distances, Blocking Sets, and Kernels for Vertex Cover

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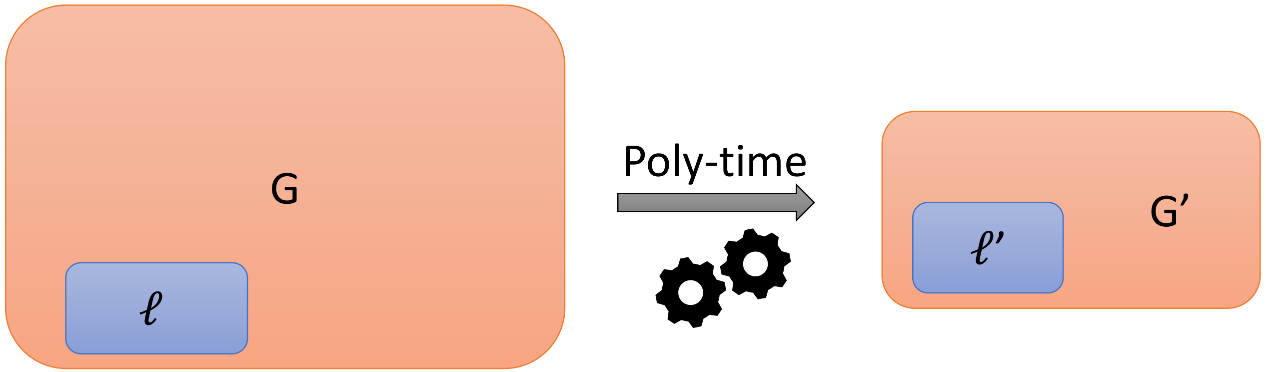
Vertex Cover Kernelization

Kernel

Polynomial time algorithm, that given graph G with parameter ℓ, asking for vertex cover of size k, outputs

G', k', ℓ' such that

- G' has a vertex cover of size $k' \Leftrightarrow G$ has a vertex cover of size k
- $|G'| \leq f(\ell), \ell' \leq f(\ell)$



Results for Vertex Cover kernelization

Parameter	#Vertices of kernel	
Solution size	$2k$	
Feedback Vertex Set	$O(\ell^3)$	[Jansen, Bodlaender, STACS 2011]
Solution size above LP	Polynomial	[Kratsch, Wahlström, FOCS 2012]
Odd Cycle Transversal	Polynomial	[Kratsch, Wahlström, FOCS 2012]
Modulator to d-quasiforest	$O(\ell^{3d+9})$	[Hols,Kratsch, IPEC 2017]
Modulator to pseudoforest	$O(\ell^{12})$	[Fomin, Strømme, WG 2016]
Modulator to degree 1 or 2	$O(\ell^5)$	[Majumdar et al., IPEC 2015]
Modulator to cluster graphs of bounded clique size	$O(\ell^d)$	[Majumdar et al., IPEC 2015]
Modulator to treedepth-η	$\ell^{2^{O(\eta^2)}}$	[Bougeret, Sau, IPEC 2017]

Results for Vertex Cover kernelization

Modulator to	#Vertices of kernel
Independent Set	$2k$
Forest	$O(\ell^3)$
$LP(G) = OPT(G)$	Polynomial
Bipartite	Polynomial
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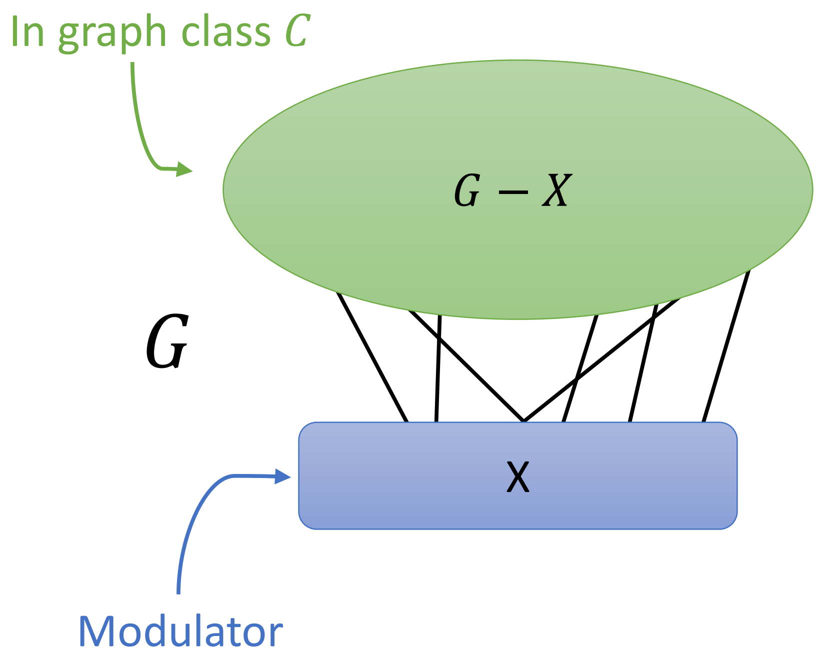

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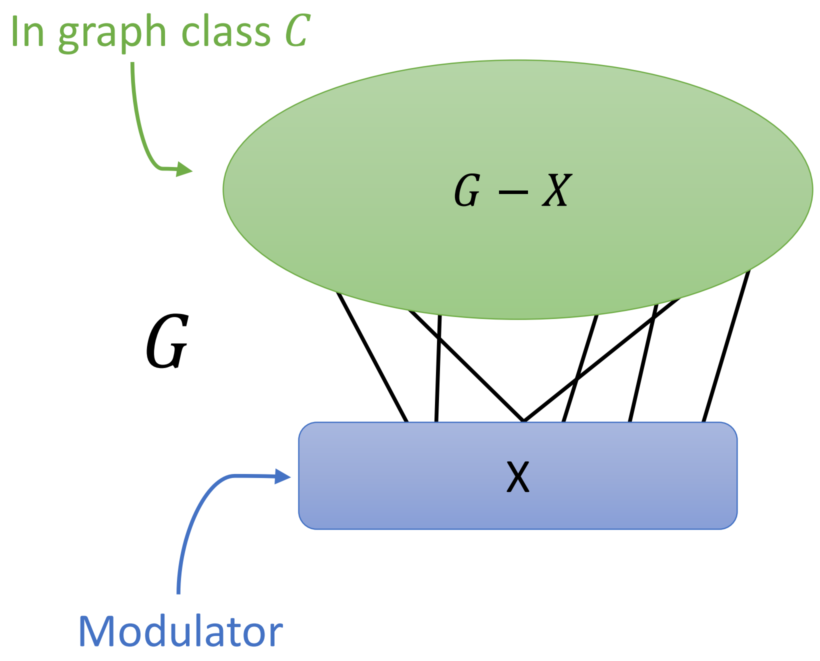
treedepth-η	$\ell 2^{O(\eta^2)}$
C	Depends on C
Elimination distance-η to C	Depends on C

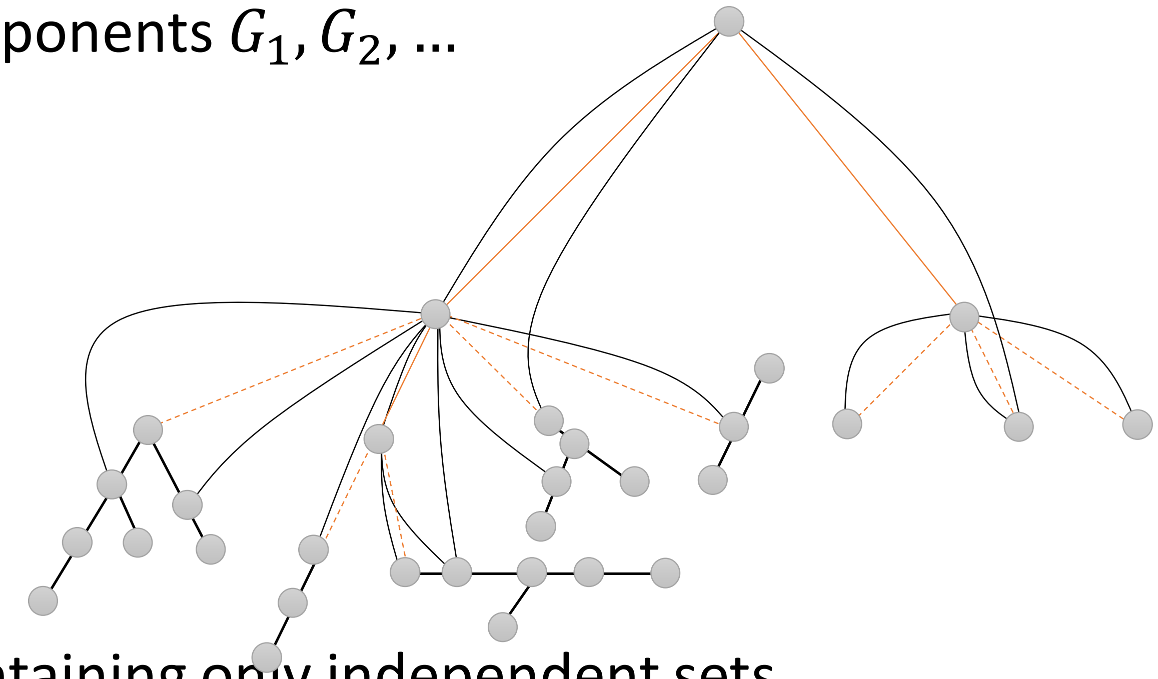


Elimination Distance

Let C be a graph class, elimination distance to C is $ed_C(G)$ is

- 0, if $G \in C$
- $\max ed_C(G_i)$ if G consists of connected components $G_1, G_2, \ldots$
- $\min_{v \in V(G)} ed_C(G - v) + 1$ otherwise

Gives corresponding elimination tree

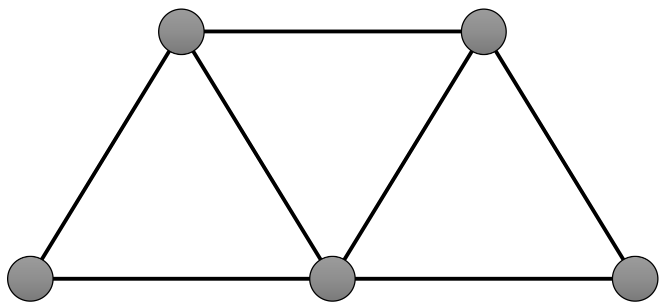


$ed_{IS}(G) = td(G) - 1$ when IS is the class containing only independent sets

Blocking sets

Subset B of $V(G)$ such that no optimal vertex cover fully contains B

- Minimal Blocking Set if B is inclusion-wise minimal
  - $\beta(G)$: size of largest minimal blocking set in G
  - $\beta(C) = \max_{G \in C} \beta(G)$ for a graph class C (possible infinite)



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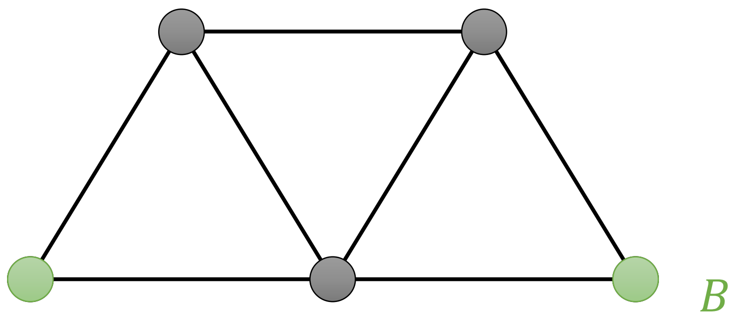

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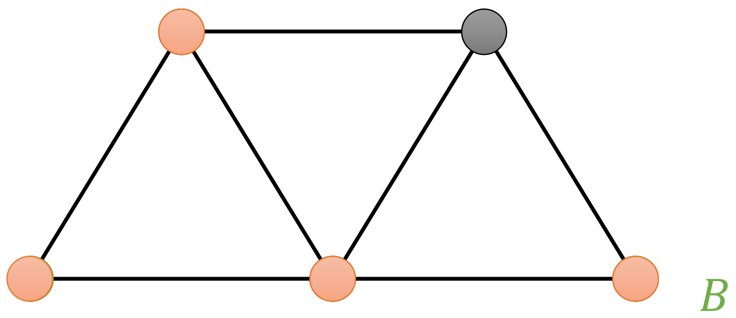

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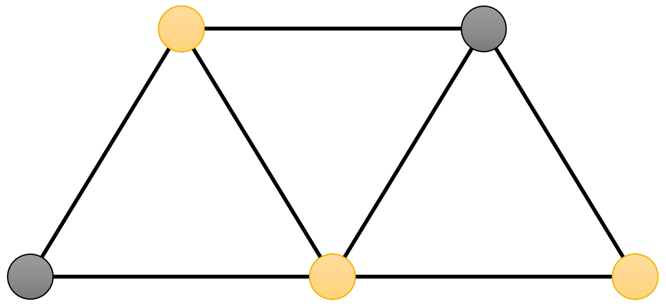

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Path	2
Size-d cliques	d
Treedepth-η graphs	$2^{\eta-2} + 1$

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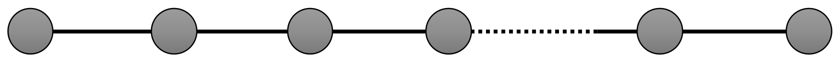

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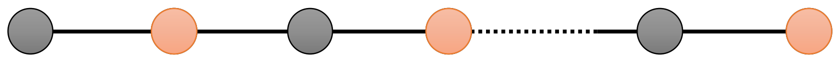

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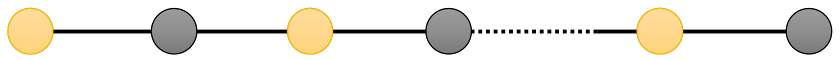

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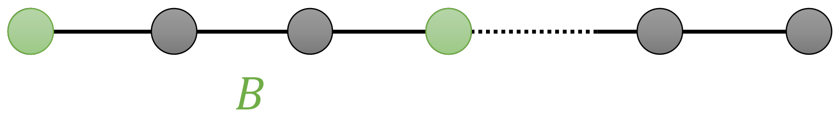

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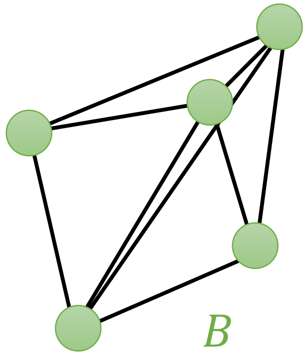


$\beta(G) = d$ for cliques

Our results

Kernelization upper and lower bounds based on blocking sets

- No kernel of size $O(|X|^{\beta(C)-\varepsilon})$
  - unless $NP \subseteq coNP/poly$, for C robust
- Efficiently reduce the number of connected components in $G - X$ to $O\left(|X|^{\beta(C)}\right)$

Blocking sets versus elimination distance

- Let C hereditary and robust, suppose G has $ed_C(G) = d$, then
  - $\beta(G) \leq (\beta(C) - 1)2^{\mathrm{d}} + 1$ if $\beta(C) \geq 2$
  - $\beta(G) \leq 2^{d-1} + 1$ if $\beta(C) = 1$
    - Tight upper and lower bound

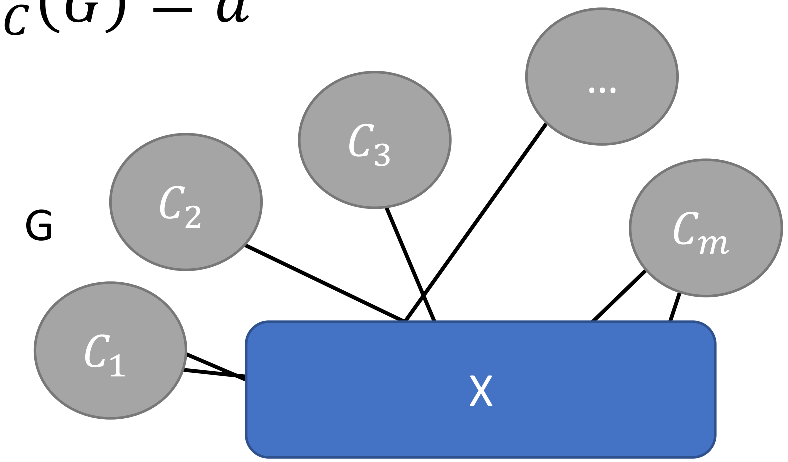
Kernelization

Consequences

If Vertex Cover parameterized by modulator to C has a polynomial kernel

- So does Vertex Cover parameterized by a modulator to $ed_C(G) = d$
  - For all d, assuming C hereditary and robust

Proof (induction)



- $d = 0$: Directly use the polynomial kernel
- $d > 0$: Polynomial kernel implies $\beta(C)$ constant
  - Implies also $\gamma = \beta(G - X)$ constant
- Reduce the number of connected components in $G - X$ to $O(|X|^\gamma)$
- For every connected components of $G - X$, add the root to X
- $G - X$ now has $ed_C(G - X) < d$
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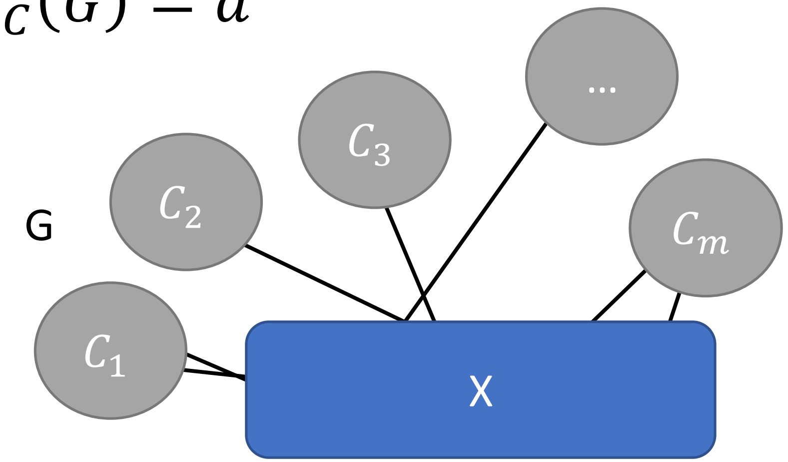

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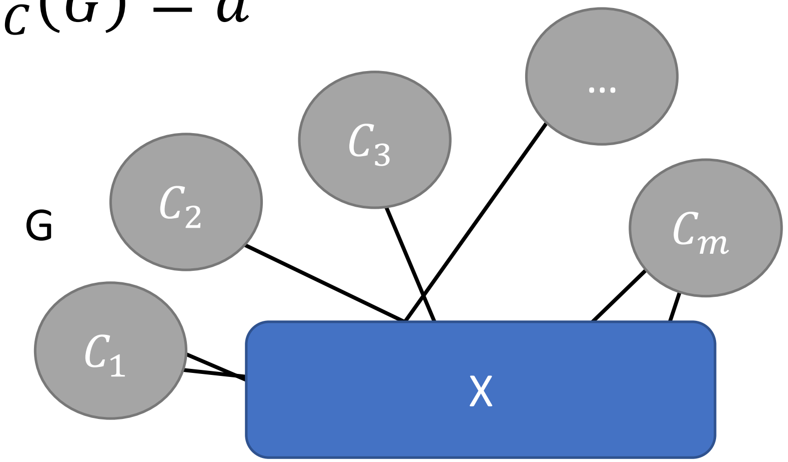

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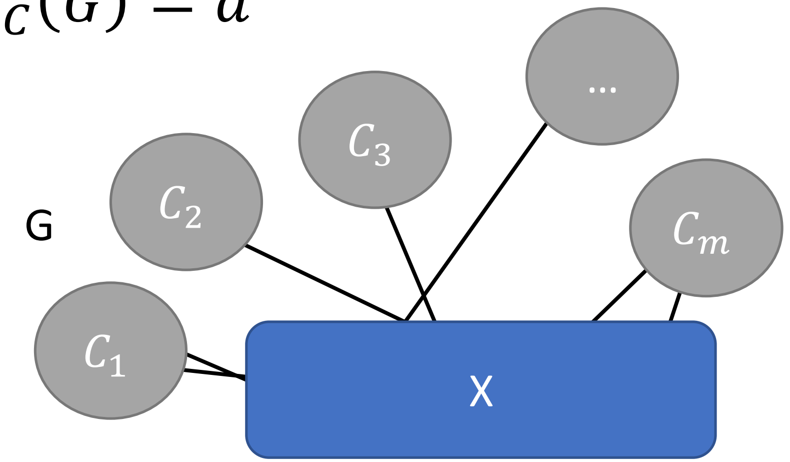

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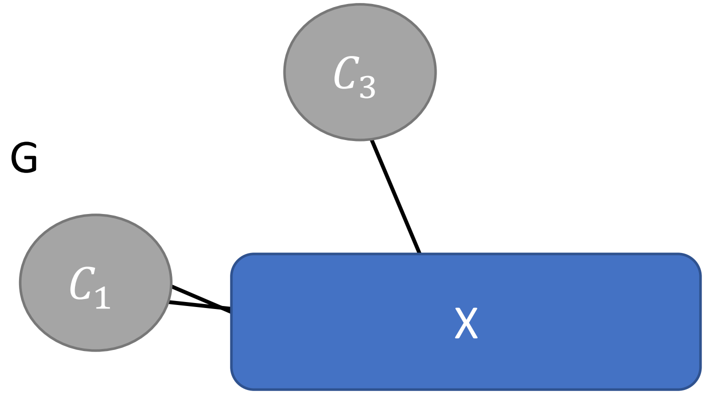

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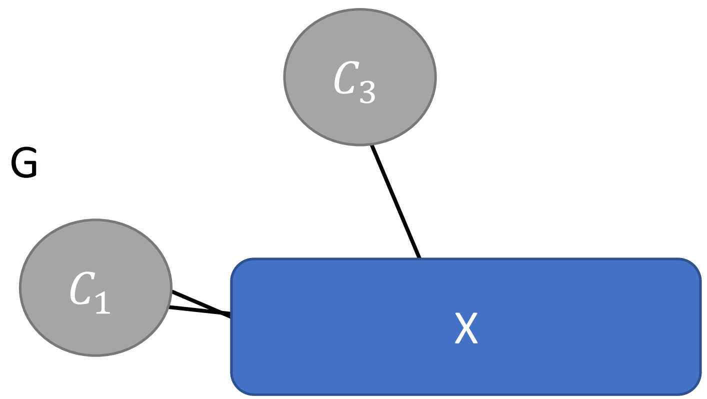

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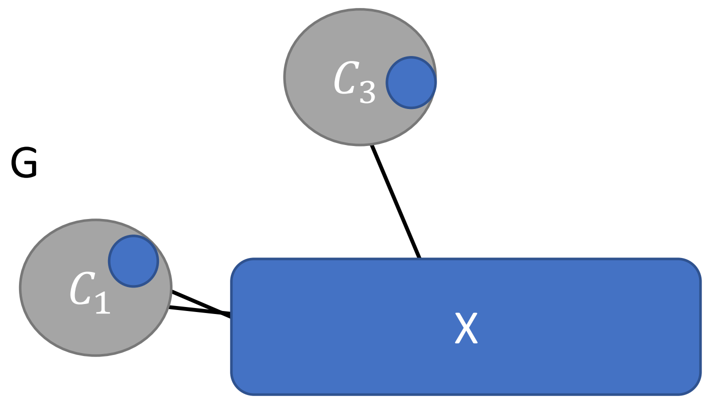

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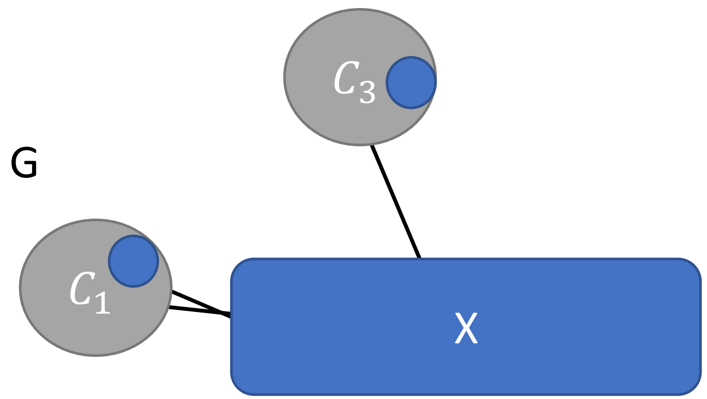

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Consequences

Generalizes existing kernels for Vertex Cover

- New parameter generalizes existing parameters!

In particular, we show that $\mathrm{OPT}(\mathrm{G}) = LP(G) + \ell$ implies that there is a modulator X of size at most $2\ell + 2$ such that $LP(G - X) = OPT(G - X)$

Reducing the number of components in $G - X$

Theorem

If $\beta(C) = d$, we can reduce the number of components in $G - X$ to $O(|X|^d)$

- Outputs G' and k' such that $OPT(G) = OPT(G') + k'$
- Runs in polynomial time (assuming C sufficiently nice)

Simple method ($O(|X|^{d+1})$ components)

- For all $S \subseteq X$ with $|S| \leq d$
  - Mark $|X| + 1$ components where $N(S)$ is blocking
- Remove unmarked components

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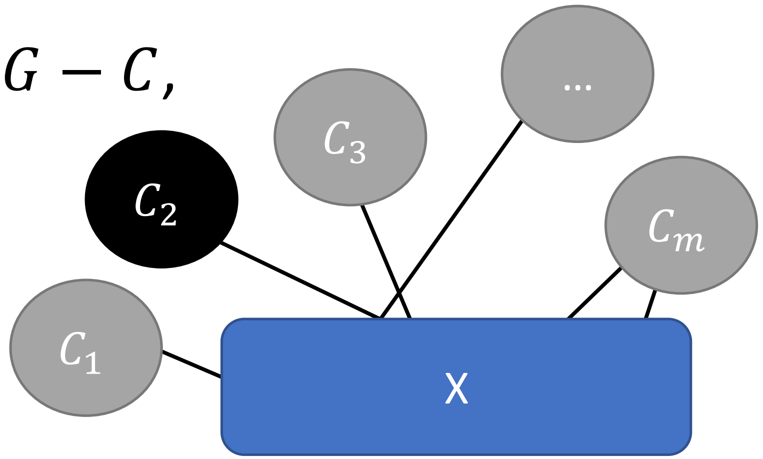
Correctness

If C unmarked, then for any minimum vertex cover Y of $G - C$,

there is a vertex cover Y' in C

- Such that $Y \cup Y'$ is a vertex cover of G
- And Y' has size $OPT(C)$



Reducing the number of components in $G - X$

Proof by contradiction

Consider $B = N(X - Y) \cap C$

- Sufficient to ensure that Y' contains B

Problem: B could be blocking

- Let $B' \subseteq B$ minimal blocking set
- Let $B'' \subseteq X - Y$ such that $B' \subseteq N(B'')$
- We marked $> |X|$ components for B''
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- Y is not a minimum vertex cover
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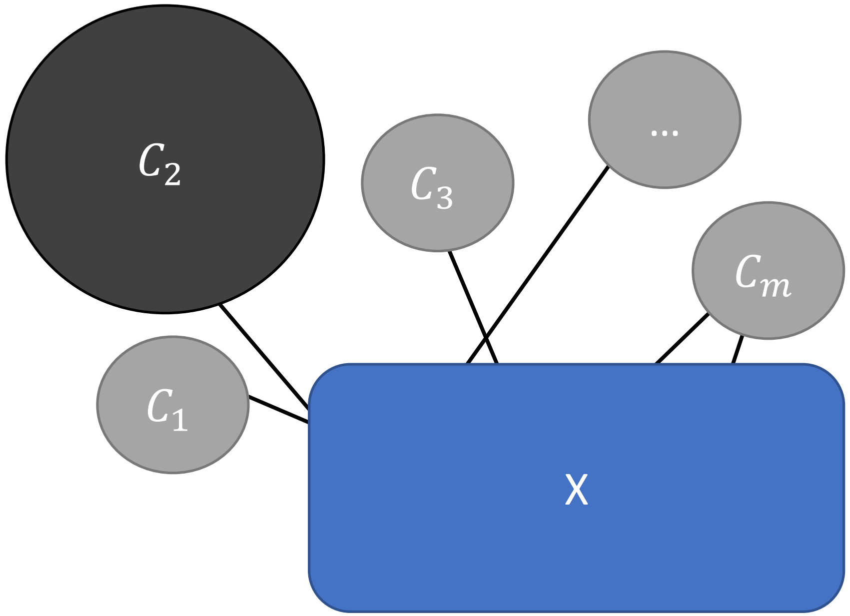

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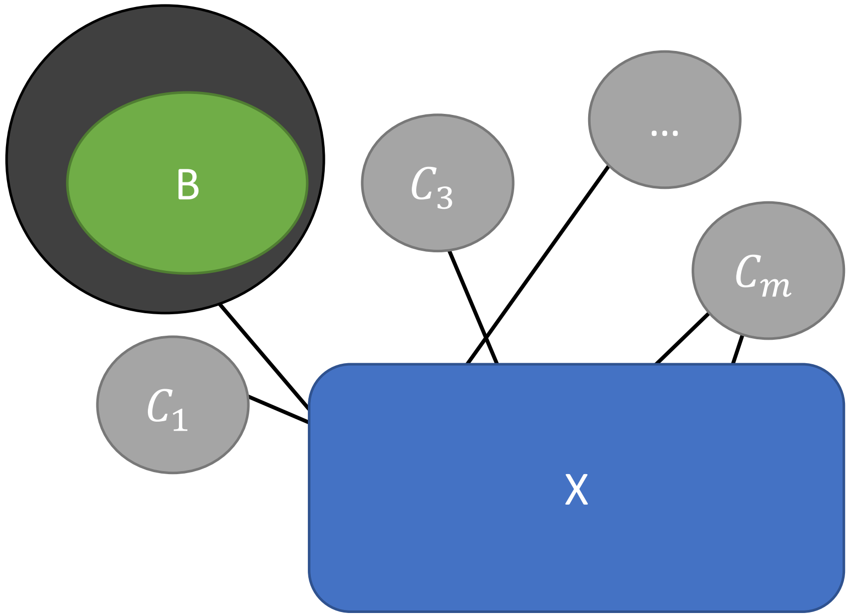

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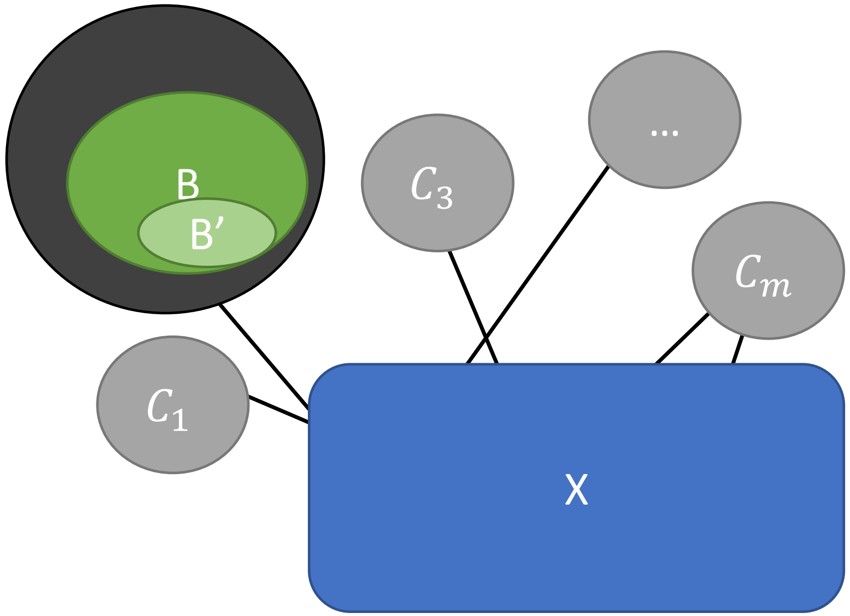

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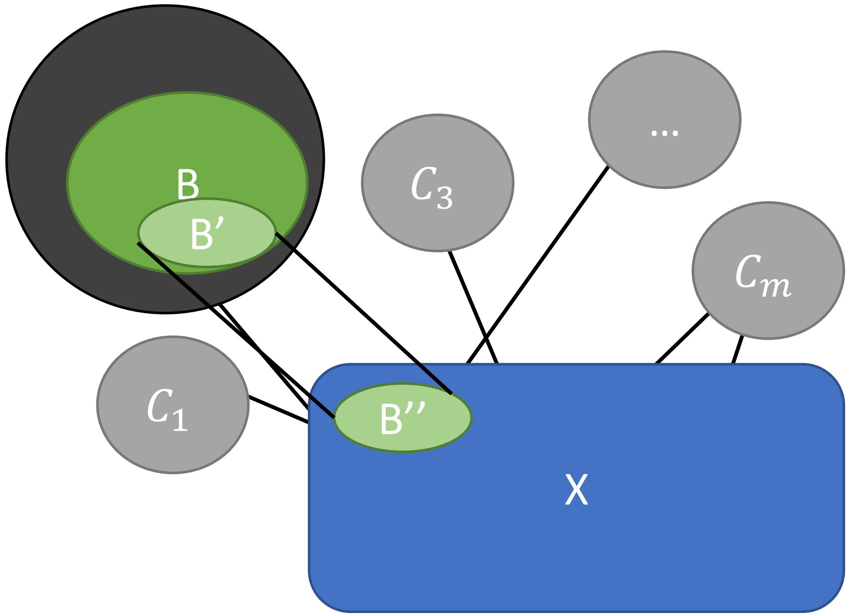

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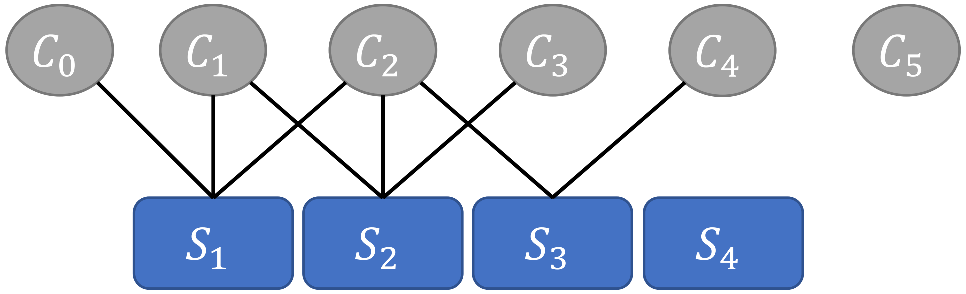
Improved method: "Crowns"

Create auxiliary bipartite graph on size-d subsets S of X and components of $G - X$

- Add connection if $N(S)$ blocking in C
- Find a maximum matching
- Remove unmatched components



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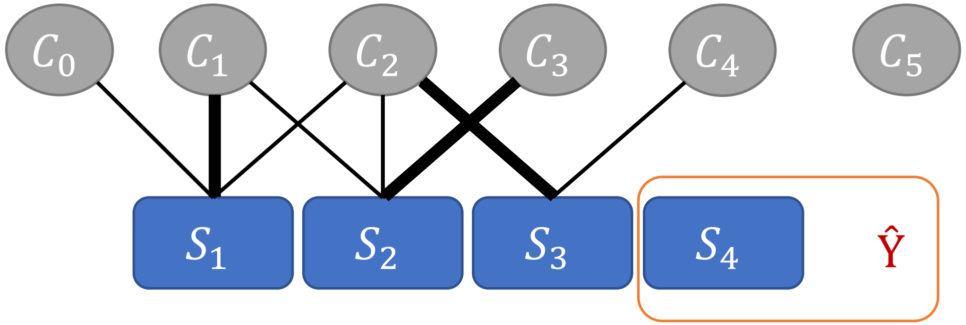


Reducing the number of components in $G - X$

Theorem

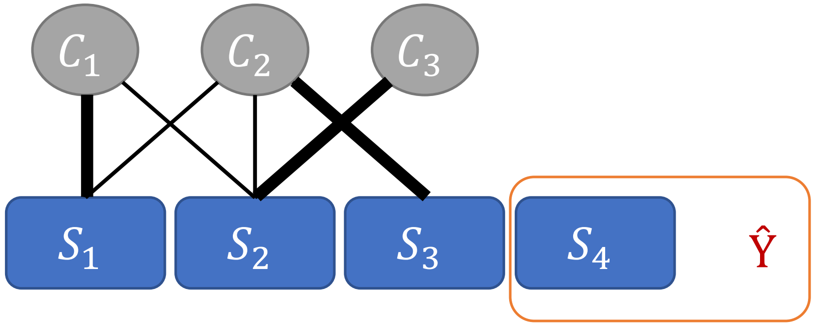
If $\beta(C) = d$, we can reduce the number of components in $G - X$ to $O(|X|^d)$

- Outputs G' and k' such that $OPT(G) = OPT(G') + k'$
- Runs in polynomial time (assuming C sufficiently nice)

Improved method: "Crowns"

Create auxiliary bipartite graph on size-d subsets S of X and components of $G - X$

- Add connection if $N(S)$ blocking in C
- Find a maximum matching
- Remove unmatched components



Blocking sets

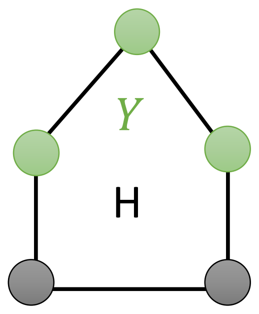
Lower bound

Let C hereditary and robust, there exists G with $ed_C(G) = d$ and

- $\beta(G) = (\beta(C) - 1)2^{\mathrm{d}} + 1$ if $\beta(C) \geq 2$
- $\beta(G) = 2^{d-1} + 1$ if $\beta(C) = 1$

We show the first result by induction

- $d = 0$: Take any graph G in C witnessing $\beta(C)$
- $d > 0$: We use the following construction
  - Size $2|Y| - 1 = 2\left((\beta(C) - 1)2^{d-1} + 1\right) - 1$
    $= (\beta(C) - 1)2^{d} + 1$



G
$ed_C(G) = d - 1$
$\beta(G) = |Y| = 3$

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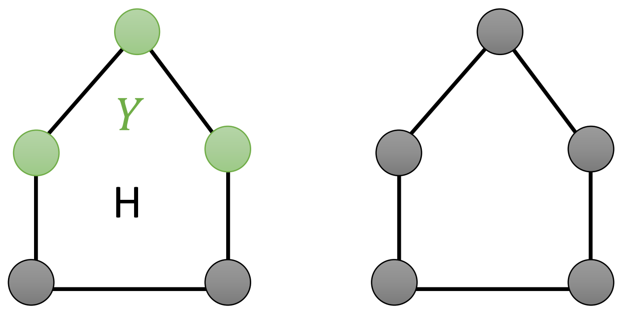

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Optimum vertex cover increases by at most one

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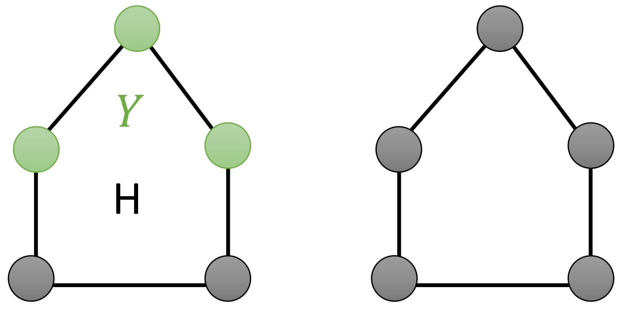

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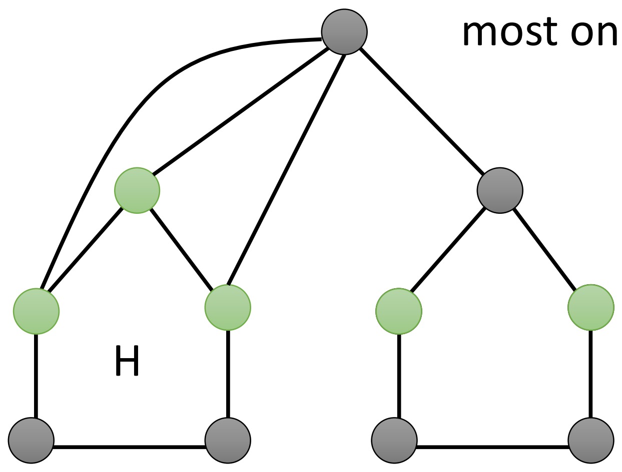

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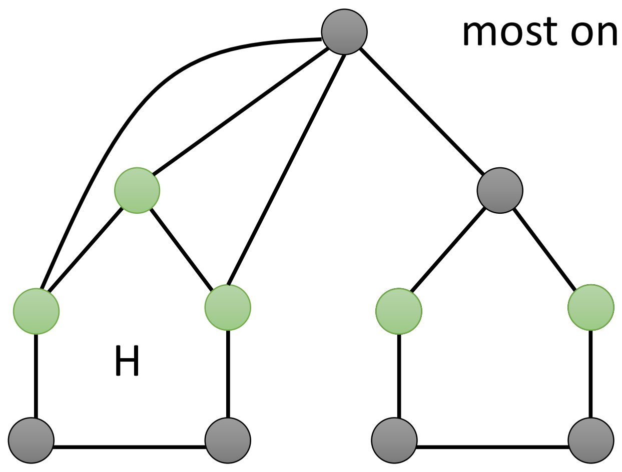

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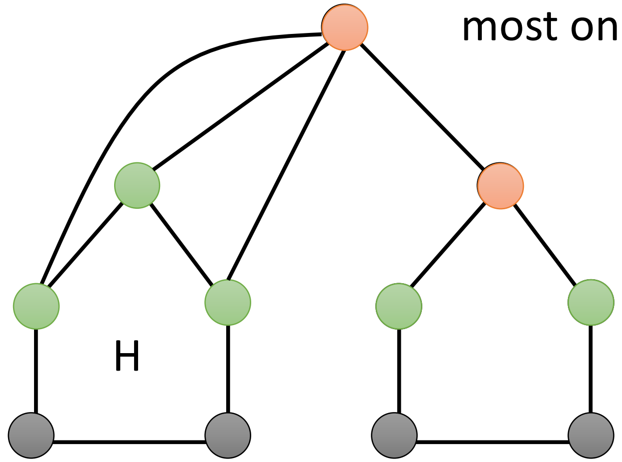

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Upper bound

Let C hereditary and robust, suppose G has $ed_C(G) = d$, then

- $\beta(G) \leq (\beta(C) - 1)2^{\mathrm{d}} + 1$ if $\beta(C) \geq 2$
- $\beta(G) \leq 2^{d-1} + 1$ if $\beta(C) = 1$

Proof idea

- If r root of decomposition
- Minimal blocking set interacts with few components of $G - r$
  - Similar to treedepth case

Summary

Link between bounded blocking set size and polynomial kernels

- Bounded blocking set size necessary
- But not sufficient

Polynomial kernel parameterized by modulator to C implies polynomial kernel parameterized by modulator to $ed_C = d$.

- Under mild assumptions on C
- Generalizes known vertex cover kernels
- Tight analysis of blocking set size of graphs where $ed_C(G) = d$

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THANK YOU