Approximate Turing Kernels

for Problems Parameterized by Treewidth

Astrid Pieterse

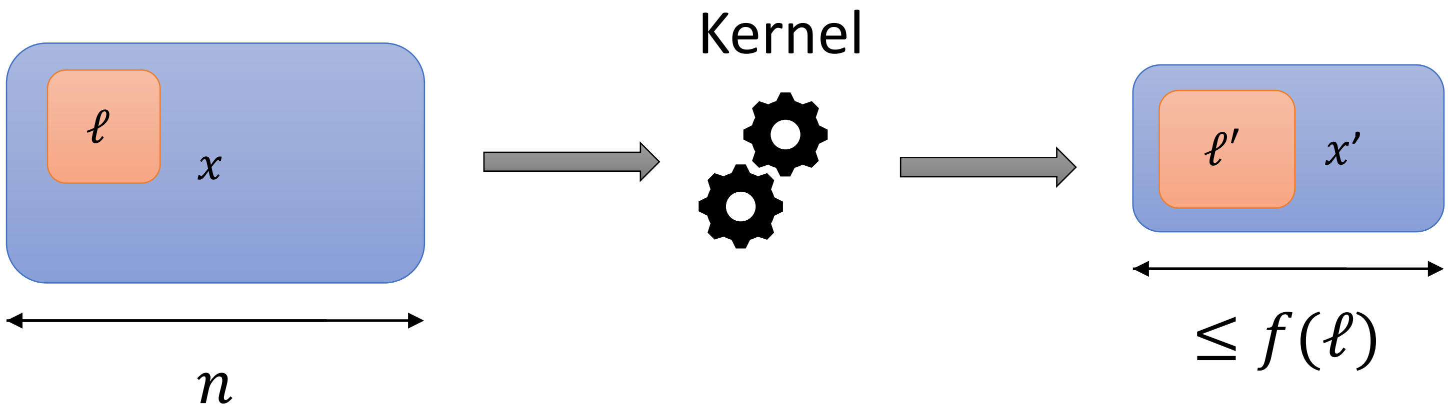
Based on joint work with

Eva-Maria C. Hols and Stefan Kratsch

https://arxiv.org/abs/2004.12683

Kernelization

Polynomial time preprocessing



Goal: obtain kernels that are small

- Every problem that is FPT has a kernel
- But only some problems have polynomial-size kernels
  - Under some complexity-theoretic assumptions

Beyond kernelization

Turing kernelization

- Allow creation of multiple instances

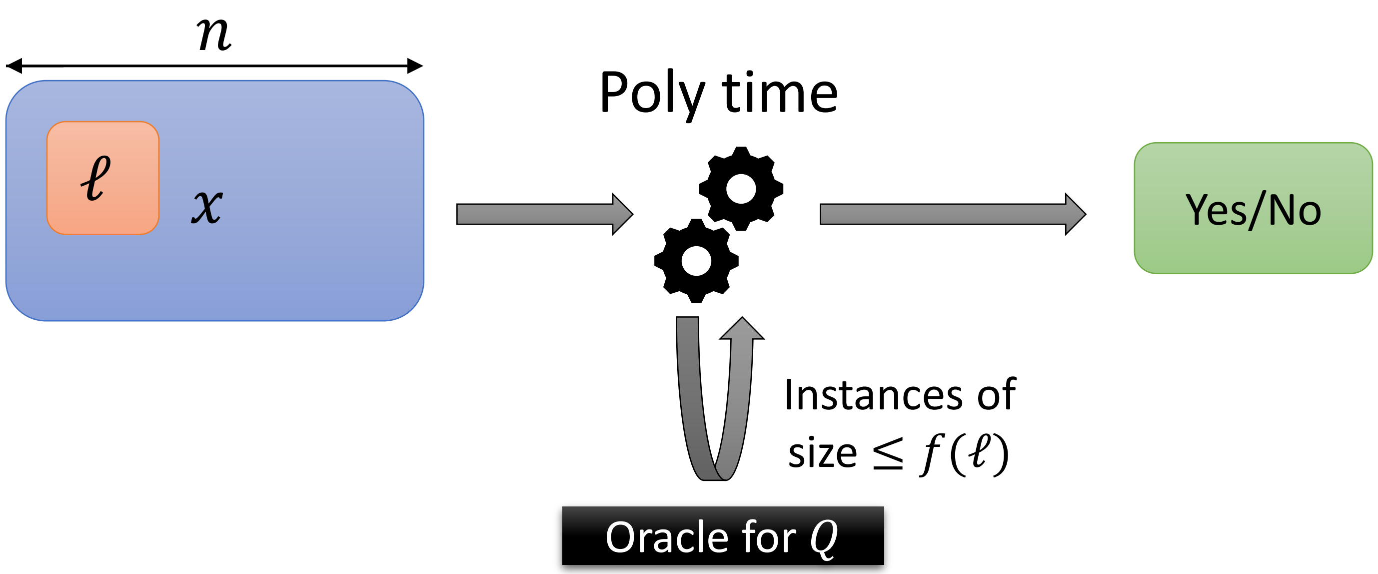
Approximate kernelization

- Relax the equivalence constraint

This talk: Approximate Turing Kernelization

Turing Kernelization

A Turing Kernel of size f for a problem Q is an algorithm that solves a given instance (x, ℓ) in time polynomial in $|x| + \ell$, when given access to an oracle that decides membership of Q for any instance with size at most $f(\ell)$ in a single step.



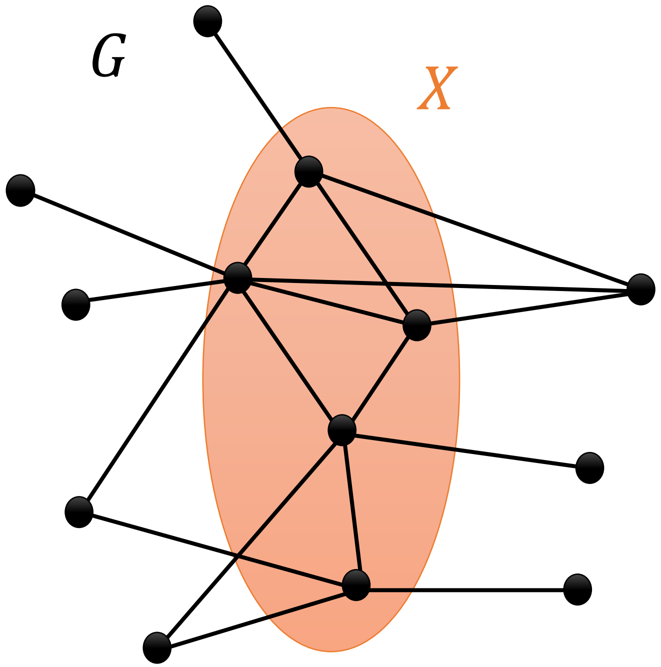
Turing Kernelization: Example

CLIQUE parameterized by vertex cover

Input Graph G, with vertex cover X of size ℓ, integer k

Question Does G have a clique of size k?

Parameter ℓ



No polynomial kernel [Bodlaender, Jansen, Kratsch 2012]

- Simple Turing kernel with $\ell + 1$ vertices

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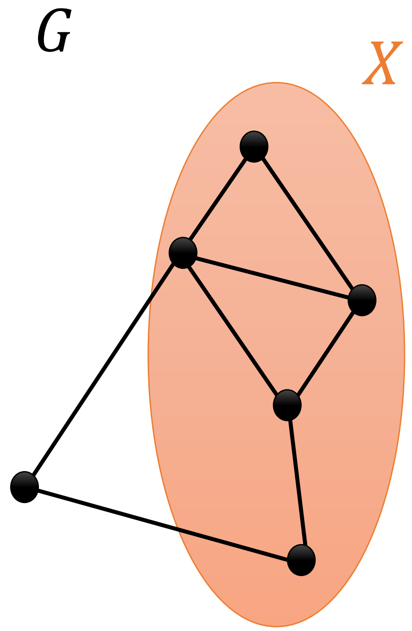

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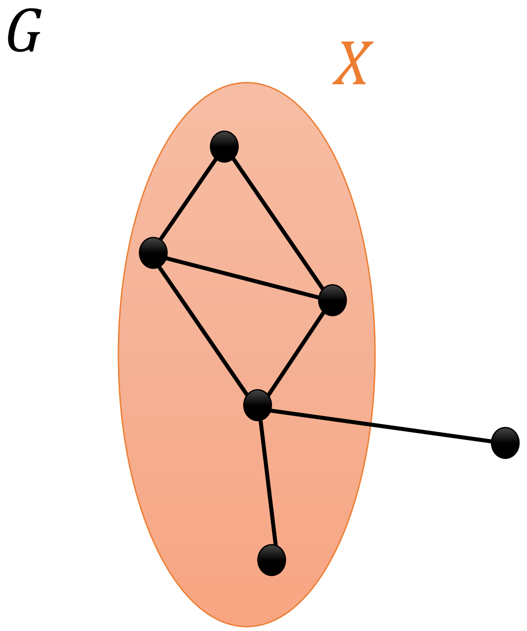

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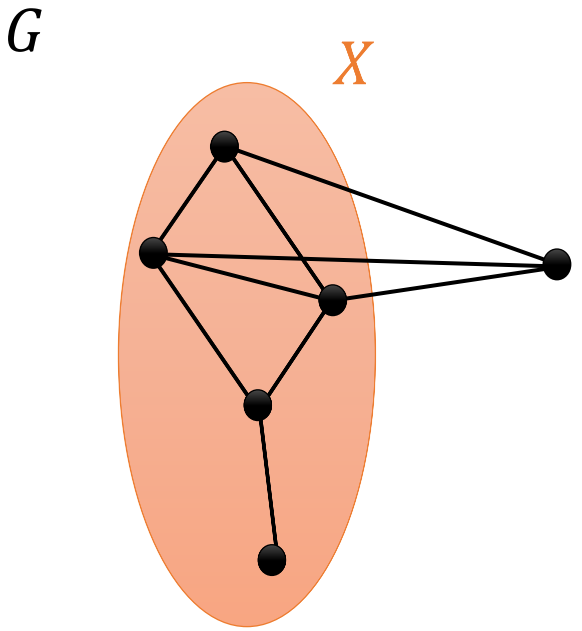
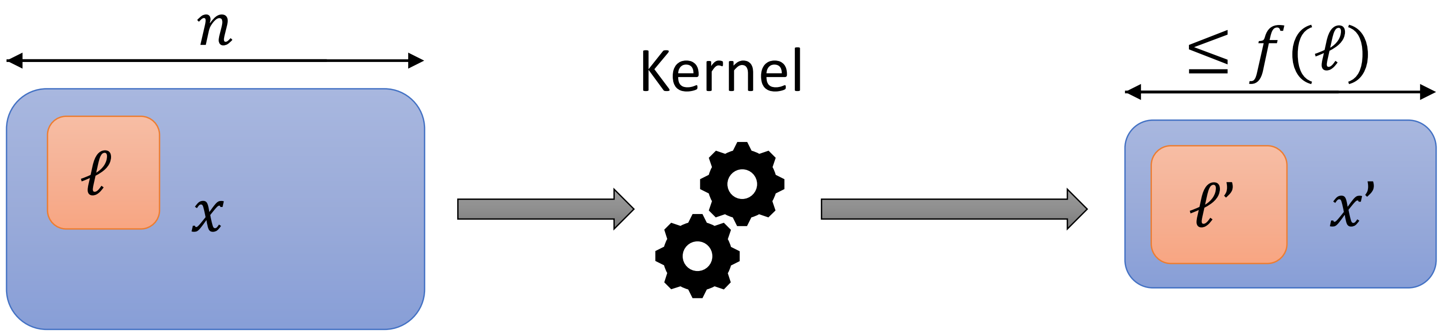


Towards approximate kernelization

Move from decision problems to optimization problems



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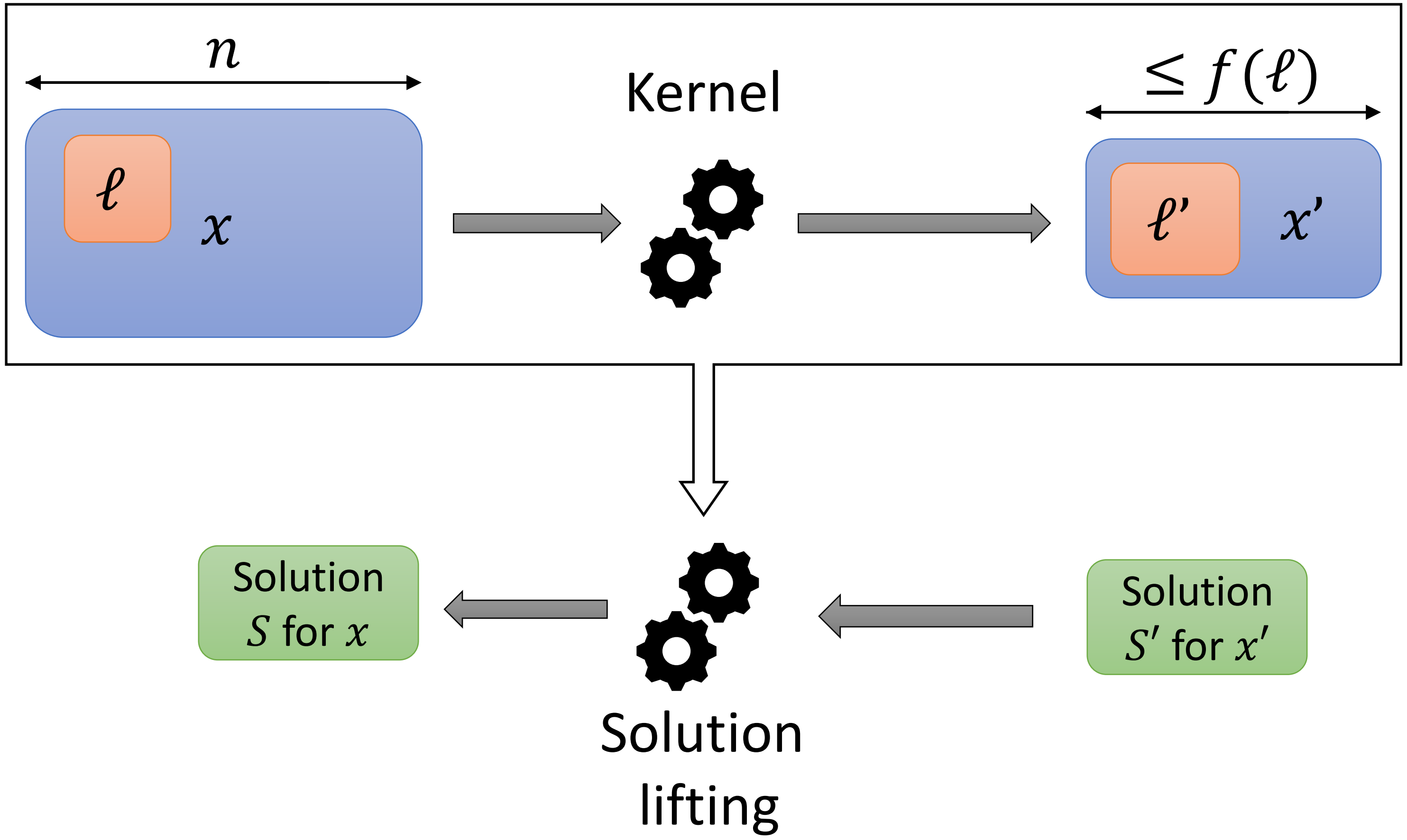

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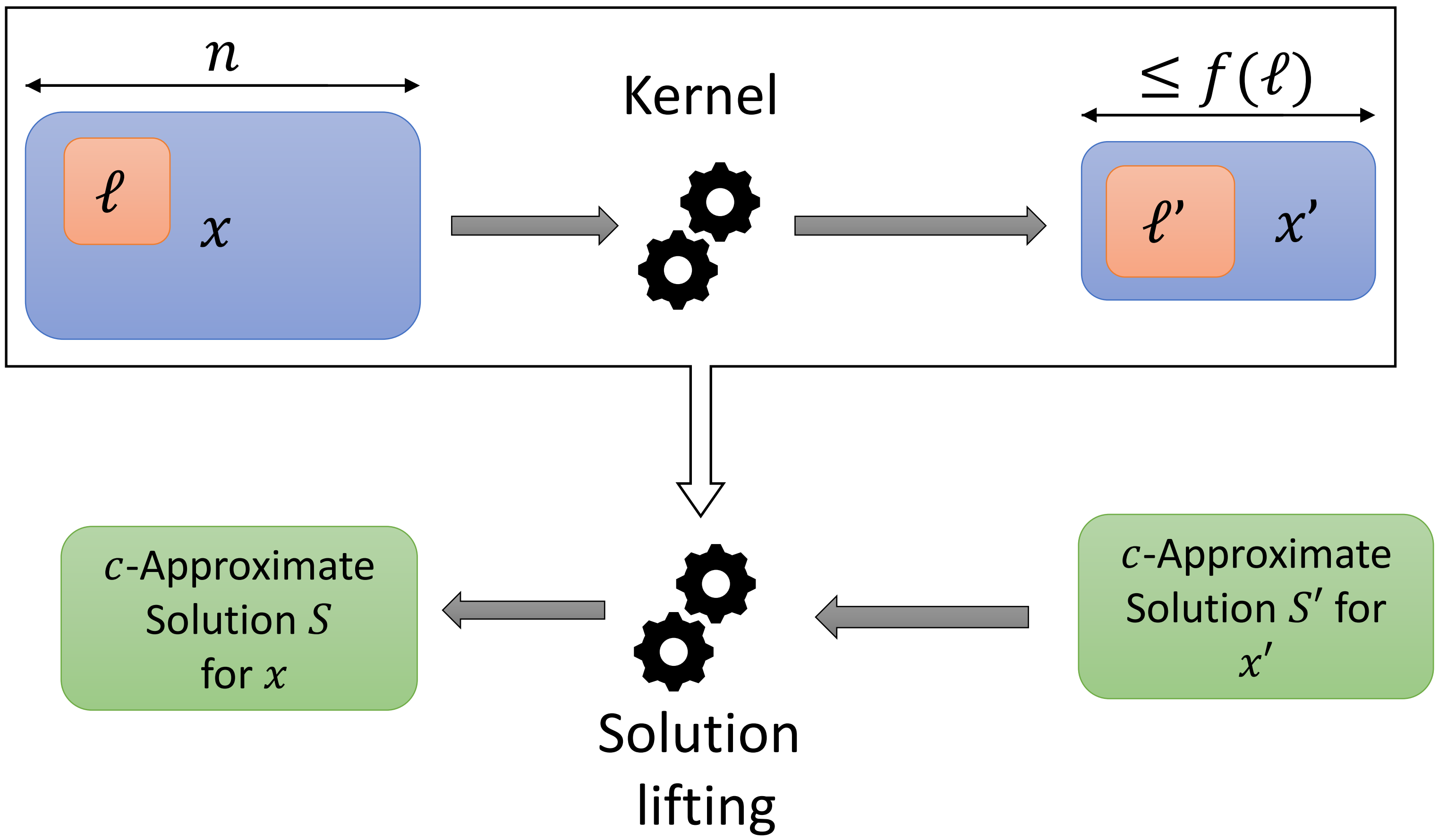

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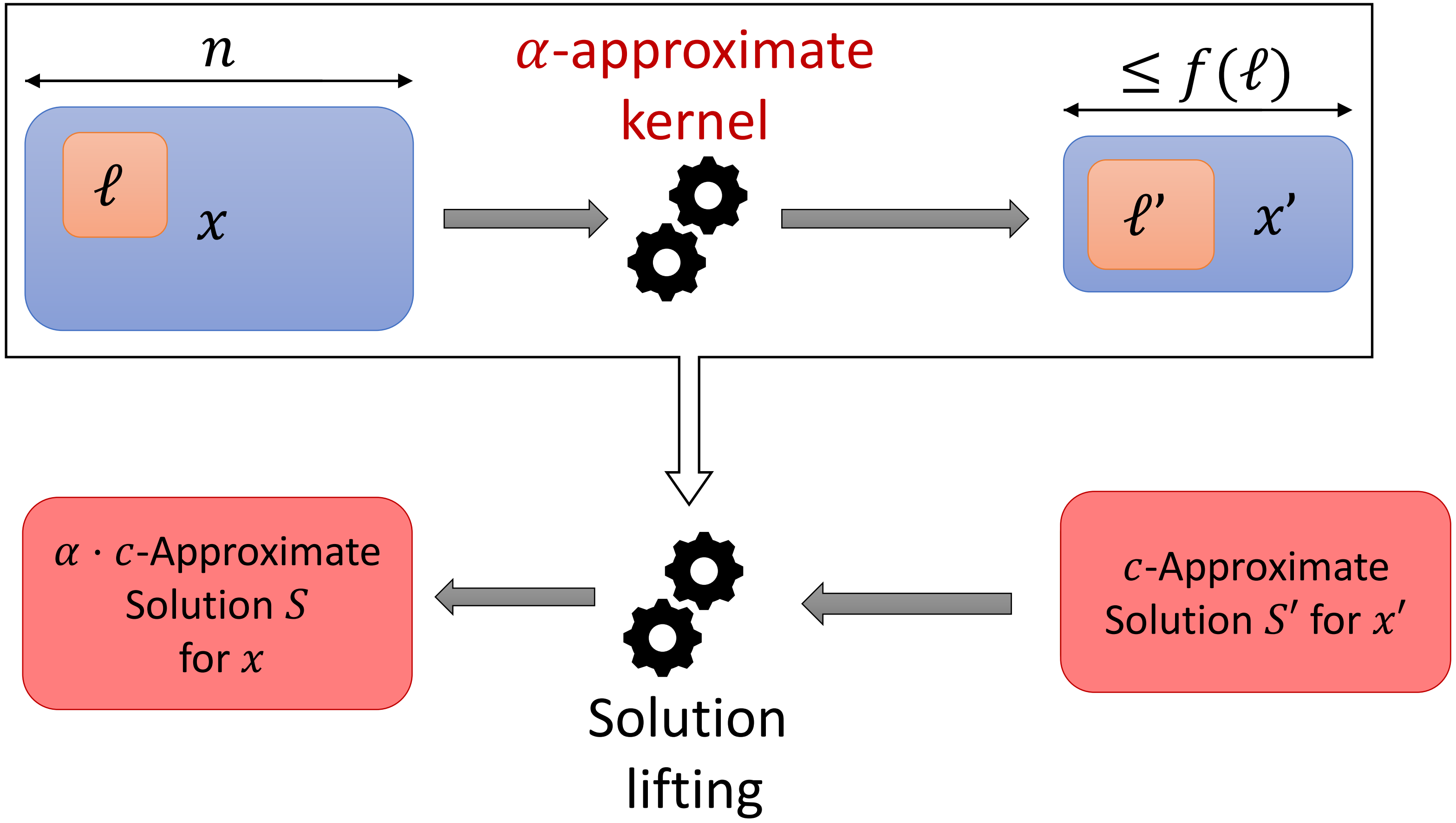

Approximate kernelization

Parameterized optimization problem Q

- Instances are pairs (x, ℓ), solutions are strings. A problem is a function Q, where $Q(x, \ell, s)$ is the value of solution s
- **Goal** find $\mathrm{OPT}_Q(x, \ell) = \min\{Q(x, \ell, s)\}$ for minimization problems

Subtlety

- If the parameter is also the optimized value, so $\ell = k$

  Vertex Cover by solution size: $Q(x, k, s) = \begin{cases} \infty & \text{if } s \text{ is not a vertex cover} \\ \min(|s|, k+1) & \text{otherwise} \end{cases}$

α-approximate kernel

Solution lifting algorithm satisfies $\frac{Q(x,\ell,s)}{OPT_Q(x,\ell)} \leq \alpha \frac{Q(x',\ell',s')}{OPT_Q(x',\ell')}$ for minimization problem

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$\max\{..\}$

Always minimum, also for maximization problems

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$-\infty$

Independent set

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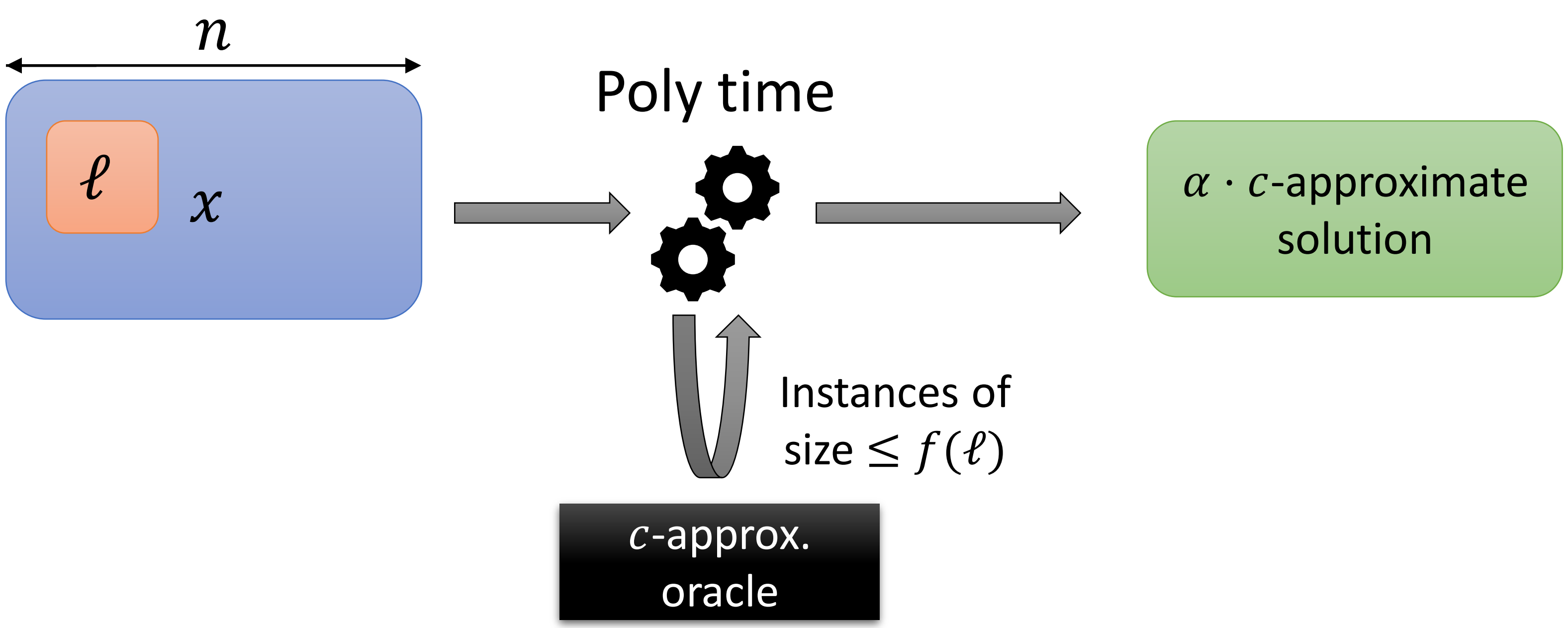
$$\frac{Q(x, \ell, s)}{OPT_Q(x, \ell)} \geq \frac{1}{\alpha} \cdot \frac{Q(x', \ell', s')}{OPT_Q(x', \ell)}$$

Approximate Turing Kernelization

α-approximate Turing Kernel

- Turing kernel, but
  - The oracle is c-approximate for some (unknown) c
  - The output must be guaranteed to be $\alpha \cdot c$-approximate



Approximate Turing Kernels, when?

When is it possible to aim for a α-approximate Turing kernel

- The problem is α-FPT-approximable
- -approximable in polynomial time

It is only useful, when

- The best-known Turing kernel is too large
  - Ideally, evidence that no polynomial Turing kernel exists
- The best-known α-approximate kernel is also large
  - Ideally, proof of nonexistence, but this seems much harder to come by

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Theorem

If a decidable problem has an α-approximate Turing kernel, it has an α-approximation algorithm that runs in FPT time.

Proof

Simply run the α-approximate Turing kernel, replacing oracle calls by calls to any algorithm solving the problem. Running time is bounded by

f(size of TK)·running time of approxTK $= f(\ell) \cdot \text{poly}(n)$

Approximate Turing Kernels, when?

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Our results

Problem	#Vertices in kernel
Independent Set	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex Cover	$O\left(\frac{\ell}{\varepsilon}\right)$
Connected Vertex Cover	$O\left(\left(\frac{\ell^2}{\varepsilon}\right)^{\left\lceil\frac{3+\varepsilon}{\varepsilon}\right\rceil}\right)$
Edge Clique Cover	$O\left(\frac{\ell^4}{\varepsilon}\right)$
Edge-Disjoint Triangle Packing	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex-Disjoint H-Packing for connected H	$O\left(\left(\frac{\ell}{\varepsilon}\right)^{\lvert V(H)\rvert-1}\right)$
Clique Cover	$O\left(\frac{\ell^4}{\varepsilon^2}\right)$
Feedback vertex Set	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$
Edge Dominating Set	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$

These problems parameterized by treewidth ℓ have $(1+\varepsilon)$-approximate Turing Kernels

- Assuming tree decomposition on input
- For all $0 < \varepsilon \leq 1$

Plus a general statement concerning "sufficiently friendly" problems

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Approximate Turing kernel checklist

Considered problems are FPT (and hence, FPT-approximable)

Polynomial kernels rare, parameterized by treewidth

- Vertex Cover and Independent Set are $MK[2]$ hard

- No good approximate kernels known
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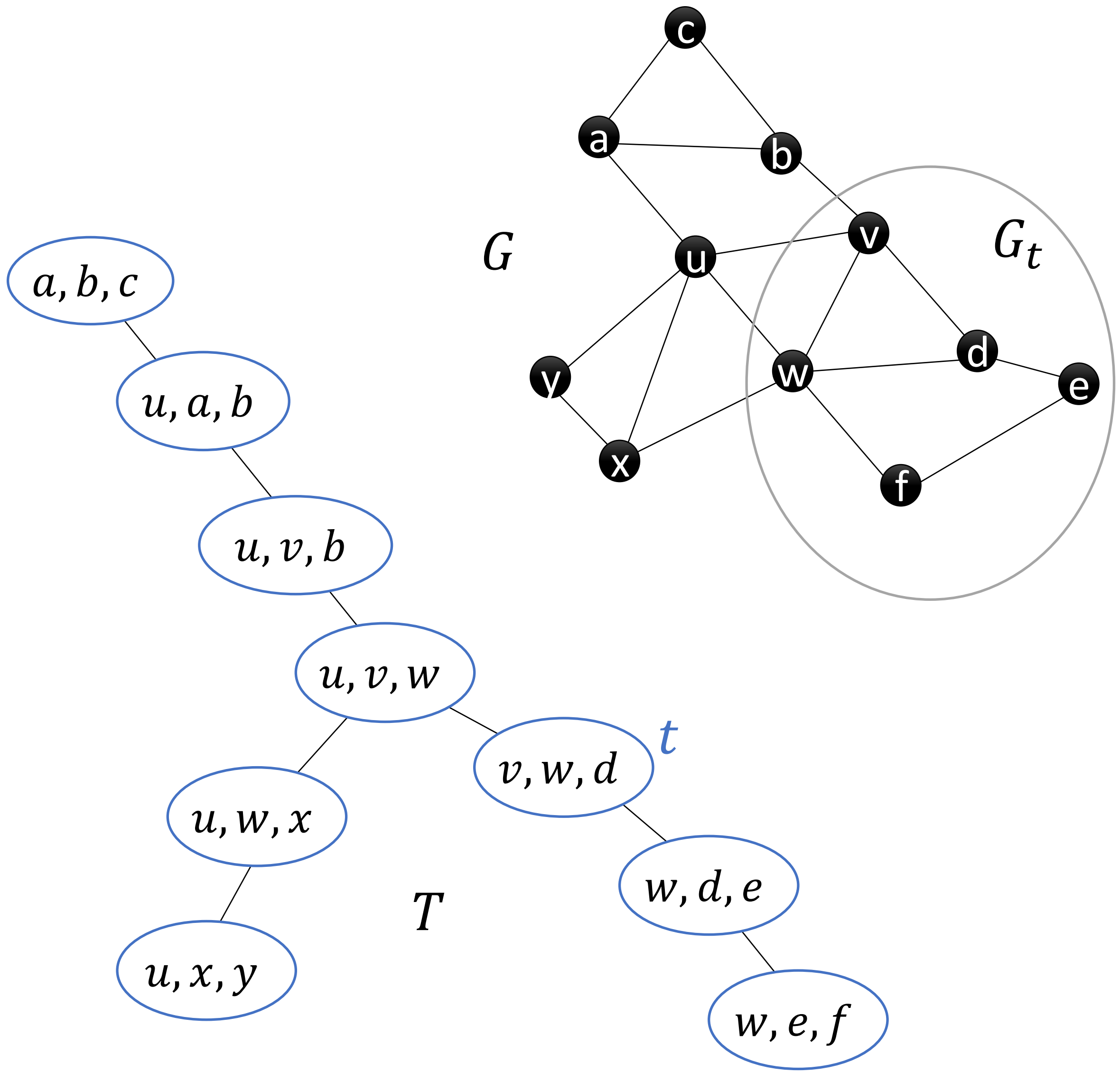
Treewidth

Tree decomposition of G

- Tree T with nodes each node t has bag $X_t \subseteq V(G)$
  - For each edge uv in G, exists bag such that $u \in X_t, v \in X_t$
  - For each $u \in V(G)$, bags in which u occurs form connected subgraph of T
  - Each $u \in V(G)$ occurs in at least one bag
- Width: size largest bag – 1



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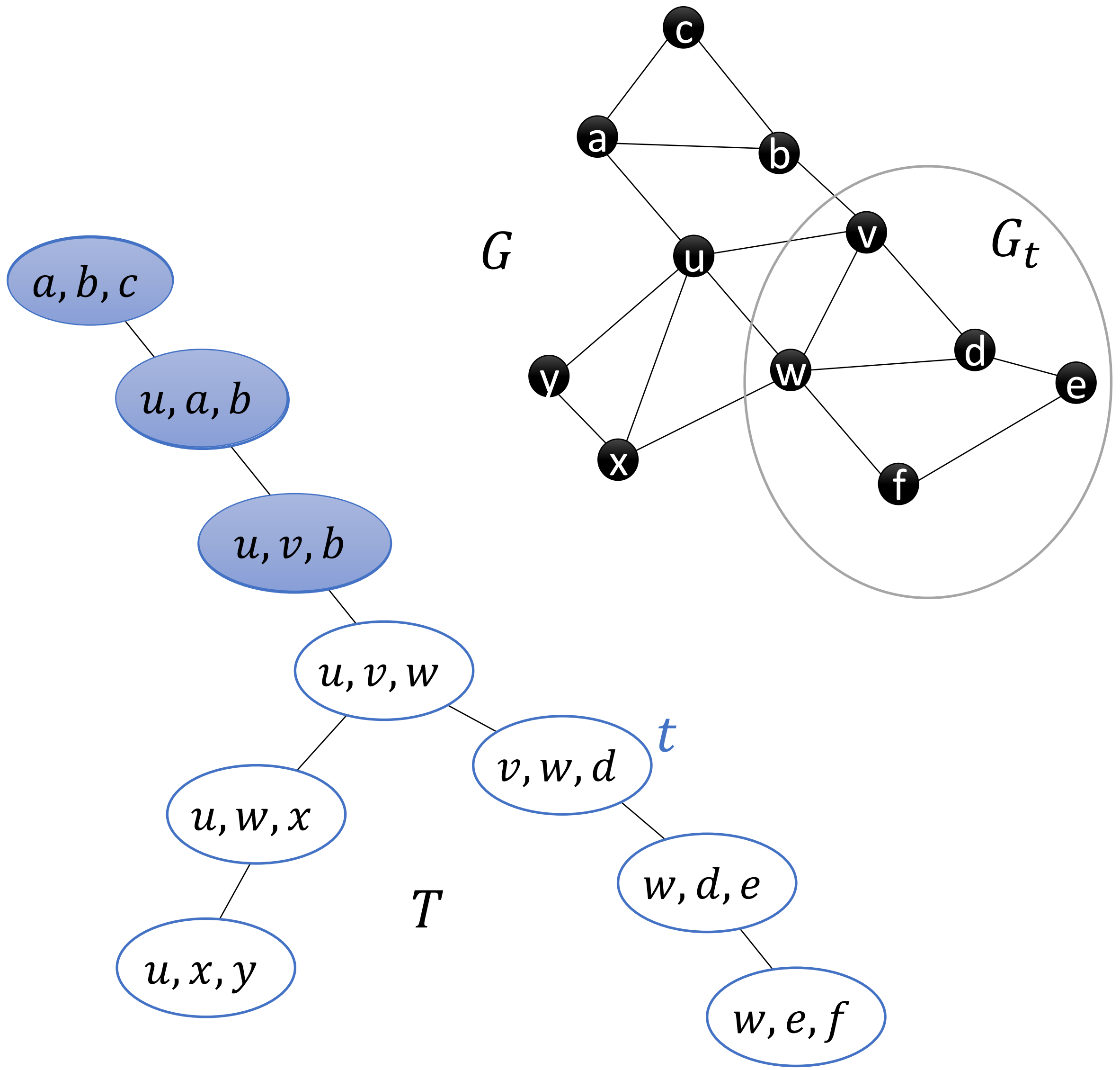

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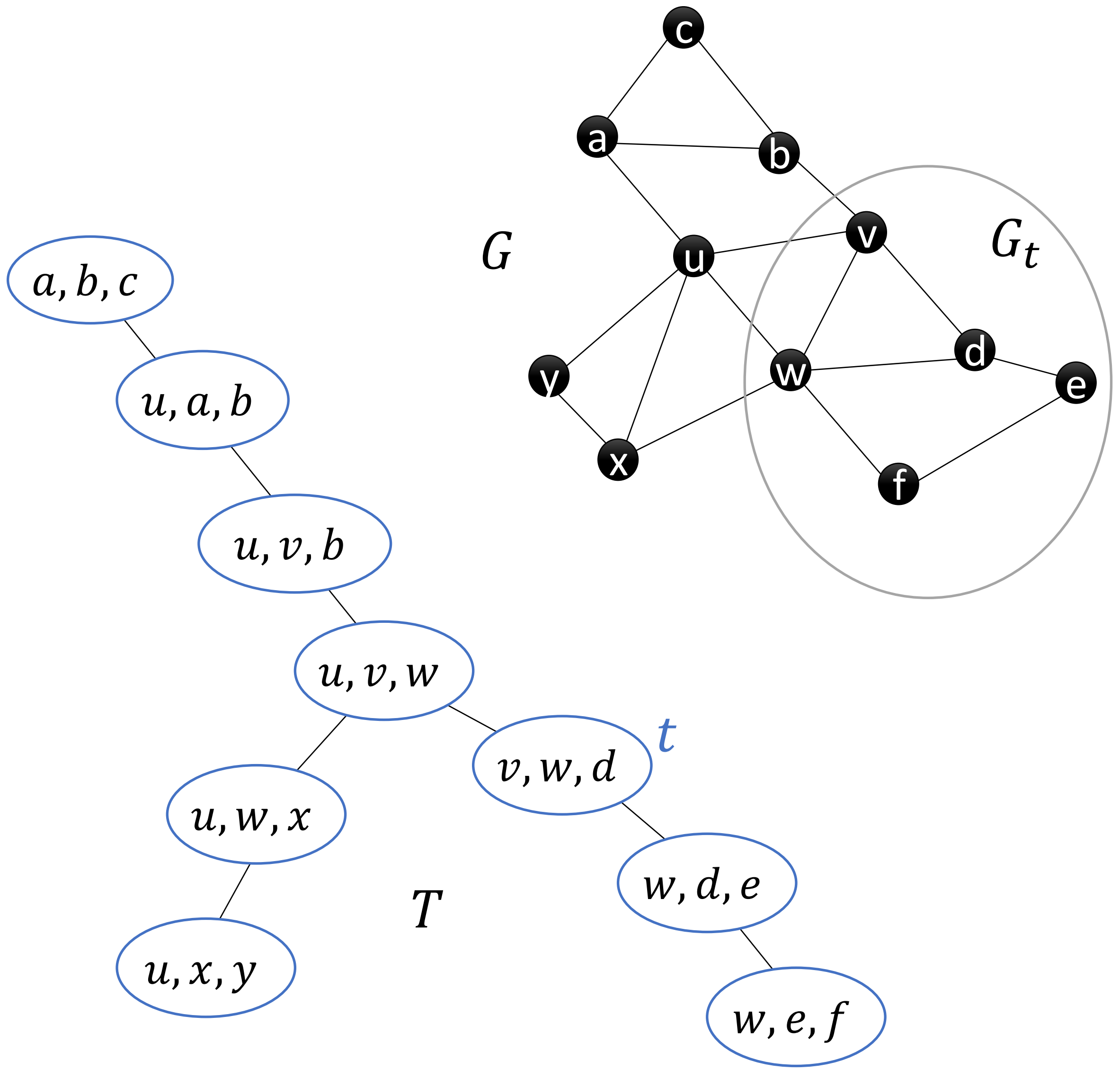

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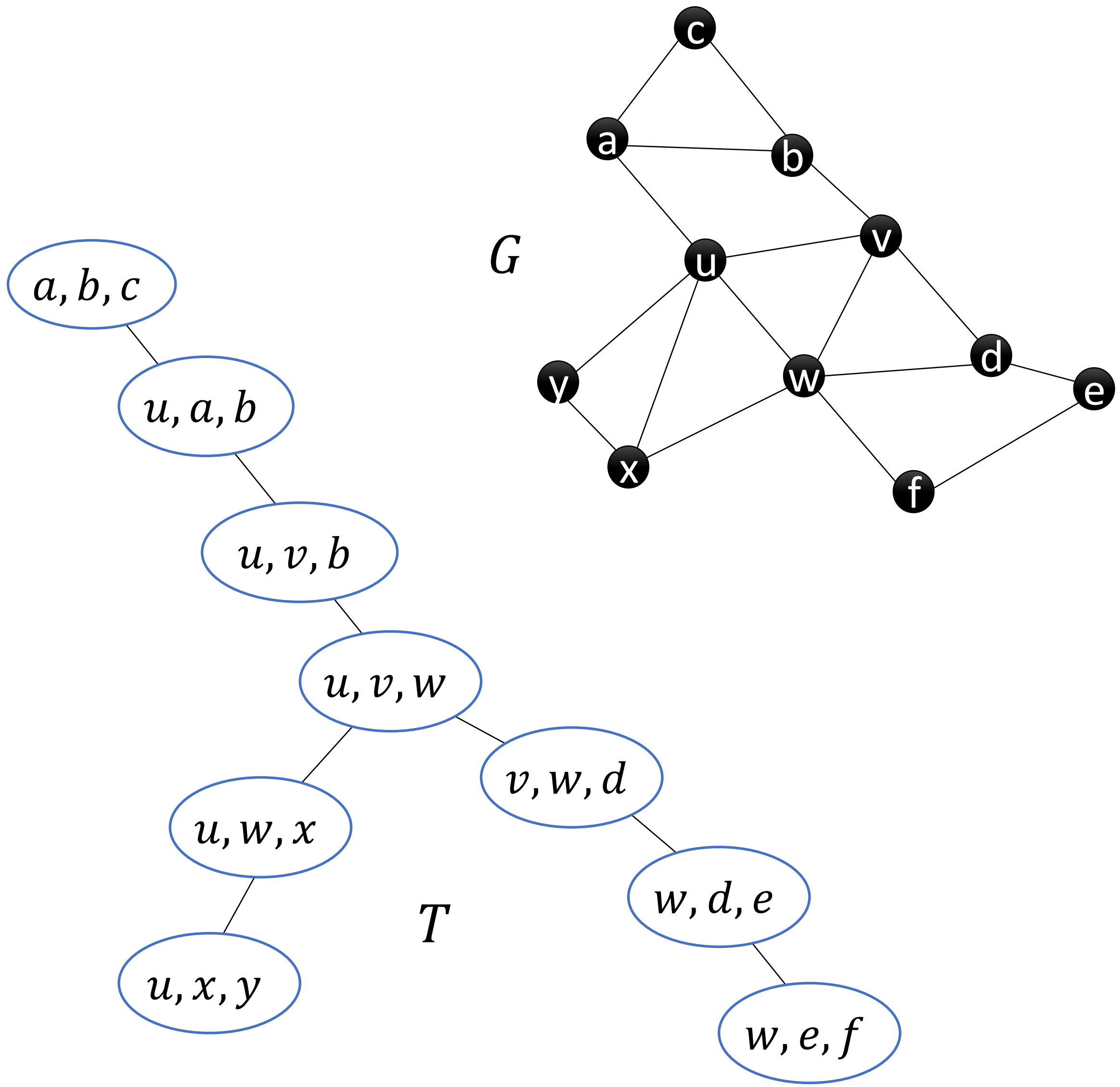

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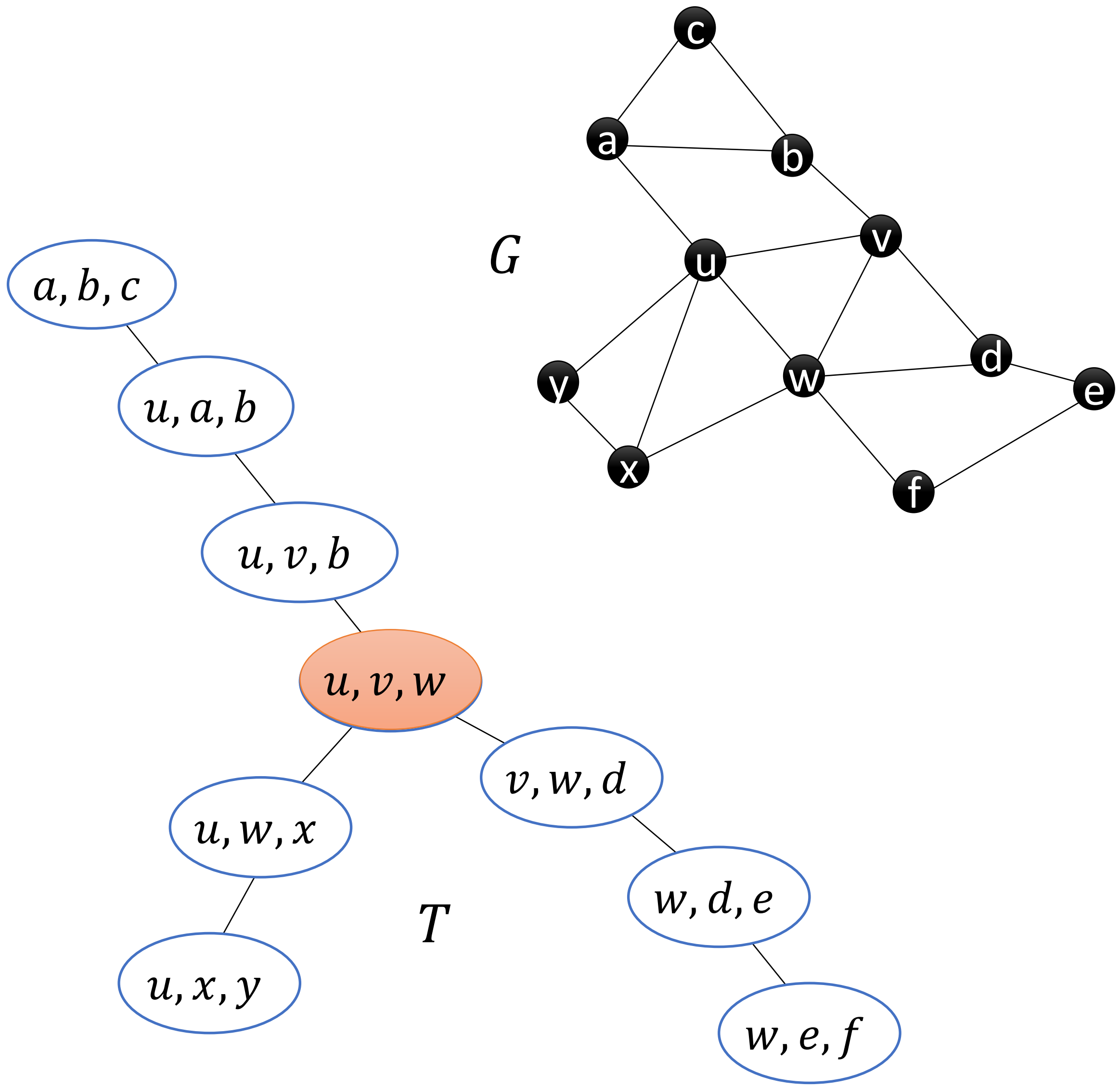

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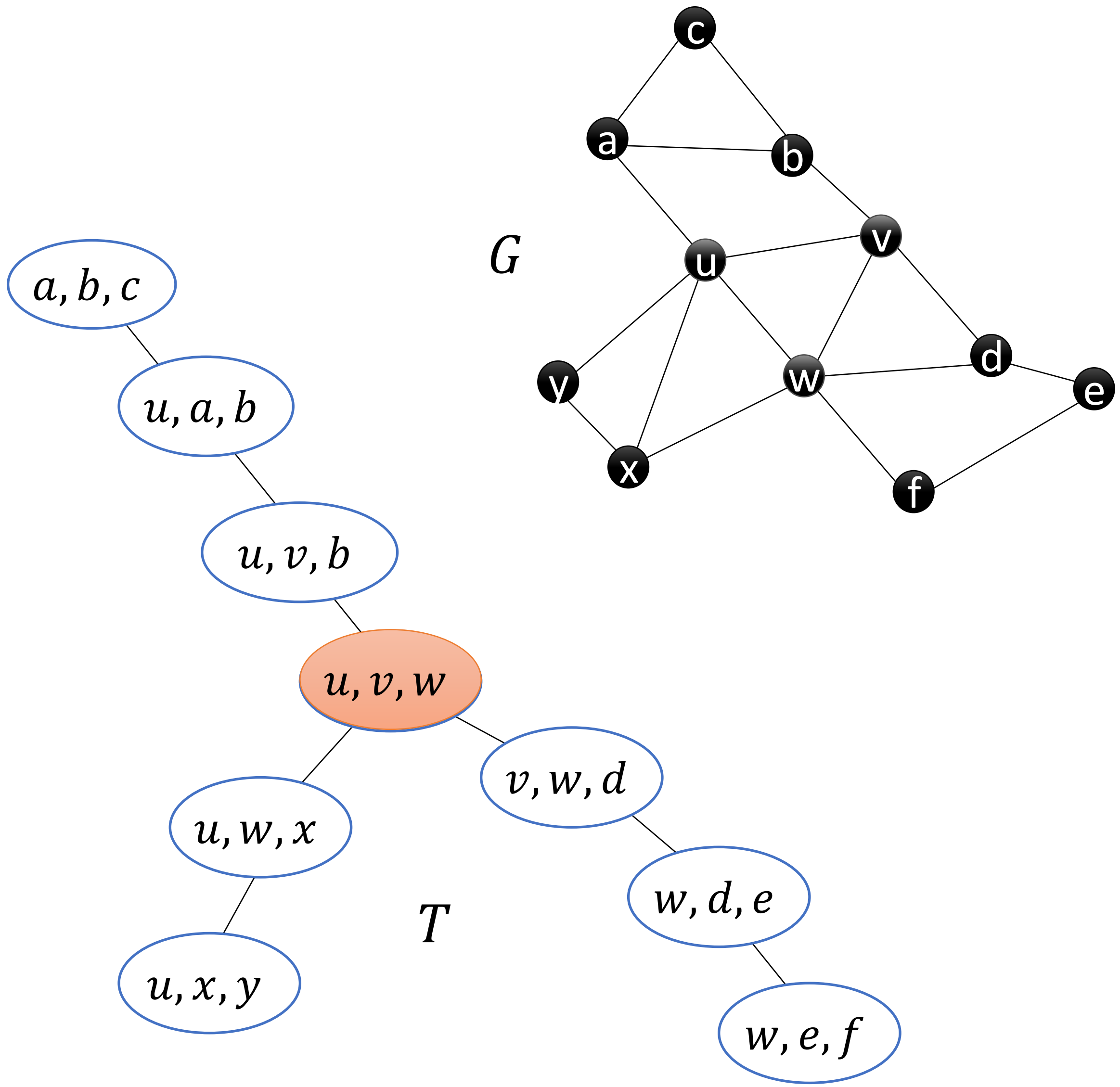

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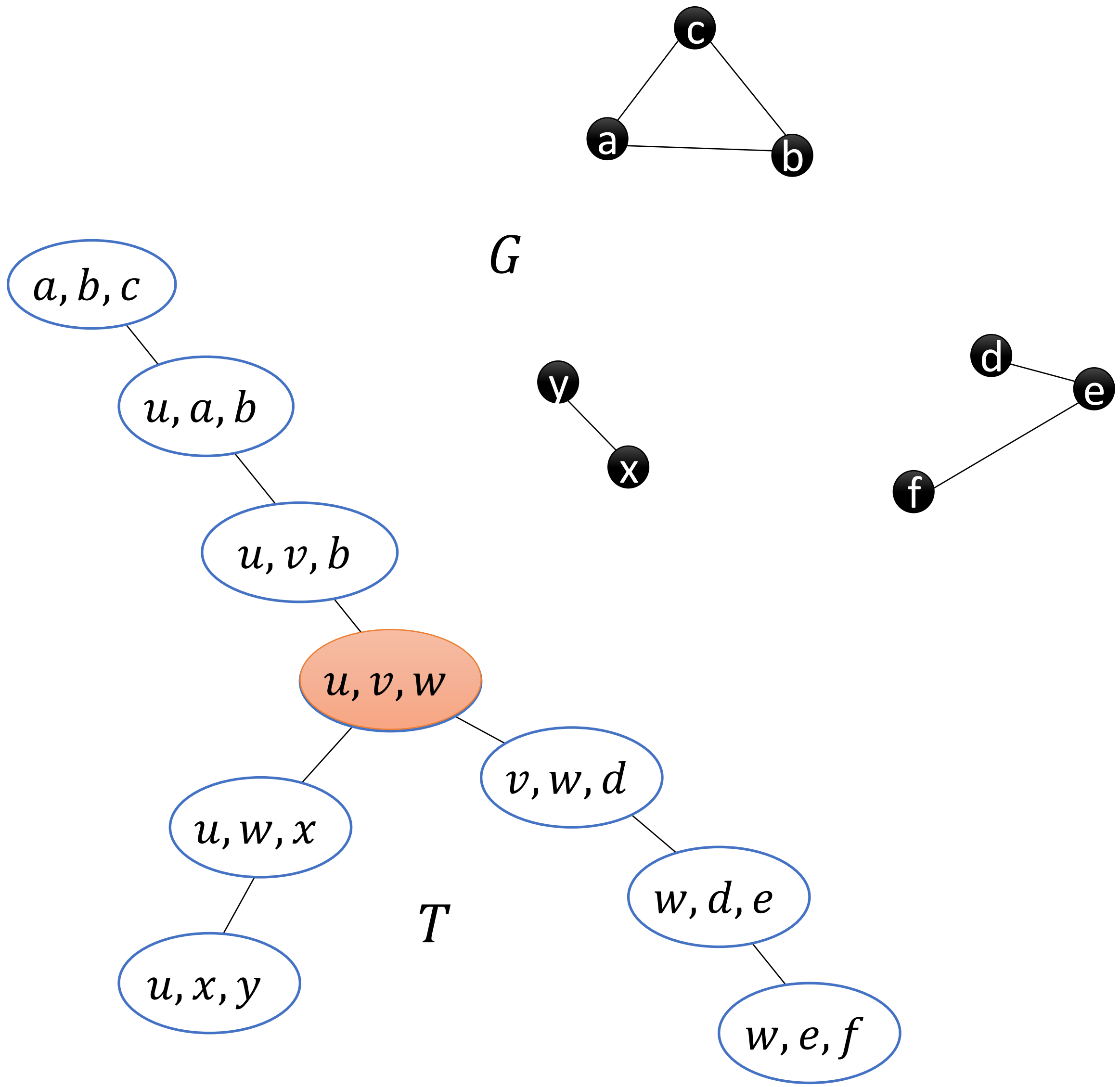

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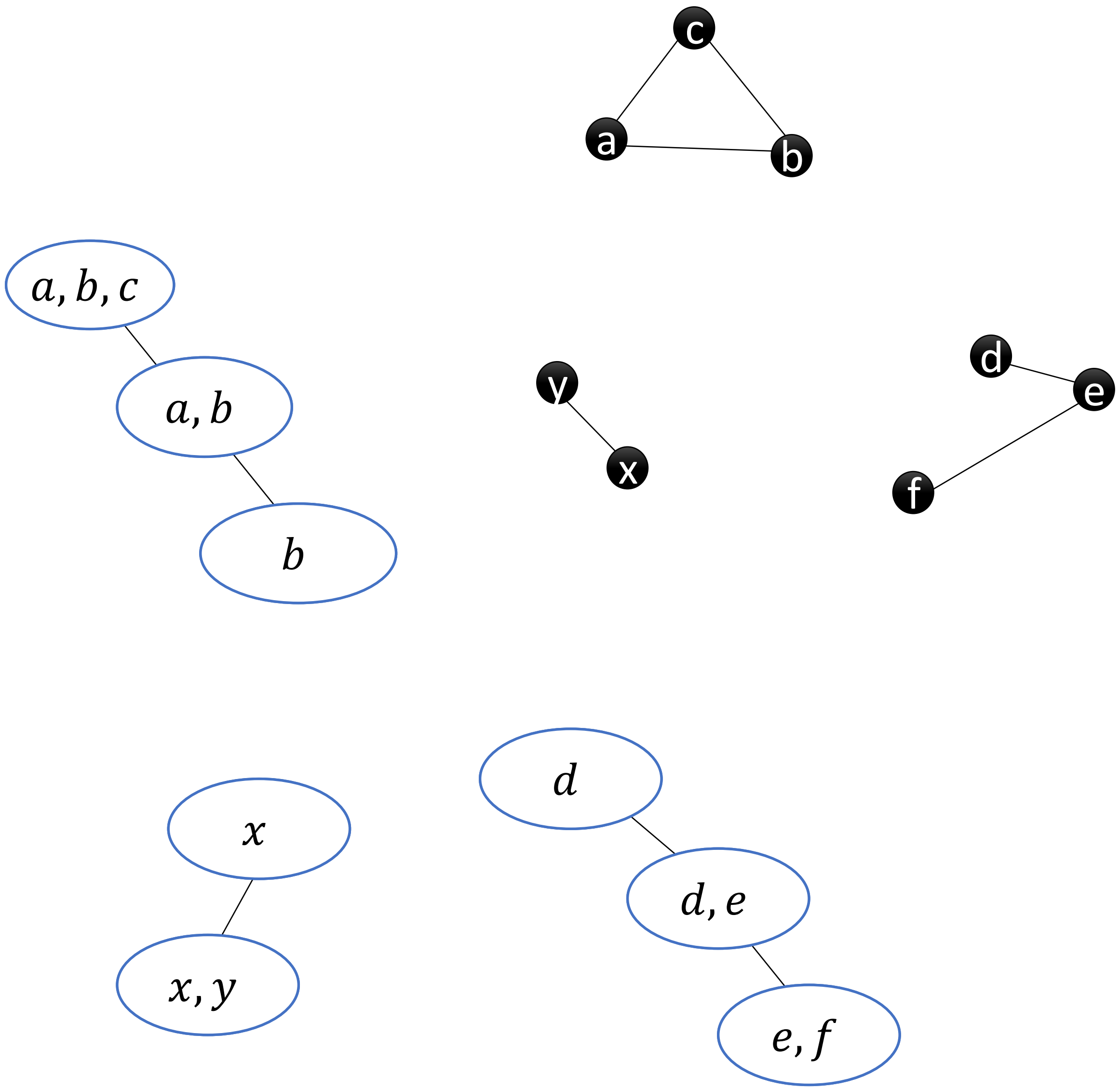


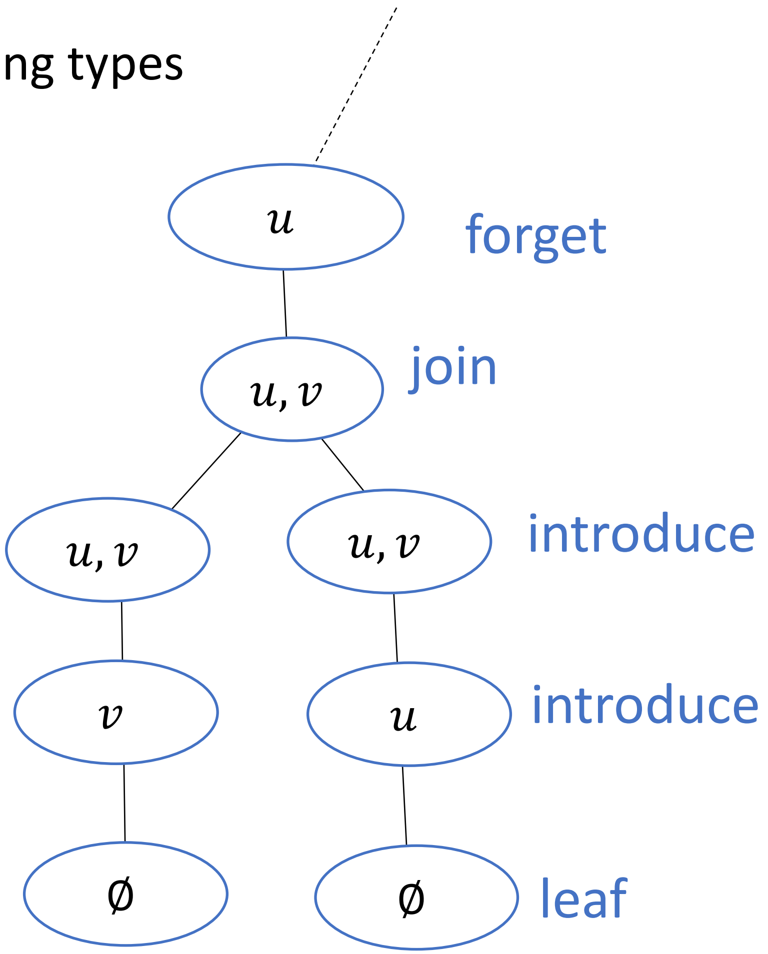
Nice tree decompositions

A rooted tree decomposition is nice if each node t is one of the following types

- **Leaf** – The node has no children, and $X_t = \emptyset$
  - $V(G_t) = V(G_t - X_t) = \emptyset$
- **Join** – The node has children t_1 and t_2 and $X_{t_1} = X_{t_2} = X_t$
  - $V(G_t) = V(G_{t_1}) \cup V(G_{t_2})$ and $V(G_{t_1}) \setminus X_t$, $V(G_{t_2}) \setminus X_t$ disjoint
- **Introduce** – The node has one child t_1 and $X_t = X_{t_1} \cup \{v\}$
  - $V(G_t) = V\left(G_{t_1}\right) \cup \{v\}$, but $V(G_{t_1}) \setminus X_t = V(G_{t_2}) \setminus X_t$
- **Forget** – The node has one child t_1 and $X_t = X_{t_1} \setminus \{v\}$
  - $V(G_t) = V\left(G_{t_1}\right)$, but $V(G_{t_1}) \setminus X_t = \{v\} \cup V(G_{t_2}) \setminus X_t$

Every tree decomposition can efficiently be made nice, without increasing its width

- We will assume $X_r = \emptyset$



Approximate Turing kernel for
Independent Set

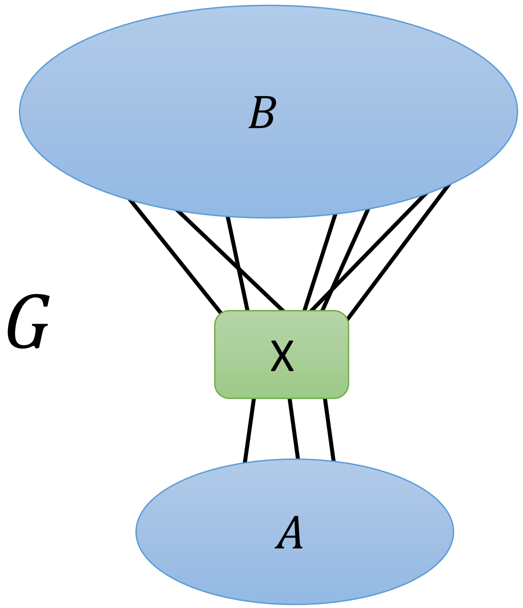
Independent Set

Theorem

Independent Set has a $(1+\varepsilon)$-approximate Turing Kernel with $O\left(\frac{\ell^2}{\varepsilon}\right)$ vertices.

Overview

1. Find a good separator X, separate the graph into (small) A and B
2. Ask the oracle for a solution S_A of part A
3. Recurse to find an approximate solution S_B for part B
4. Show $S_A \cup S_B$ is a c$(1+\varepsilon)$-approximate solution



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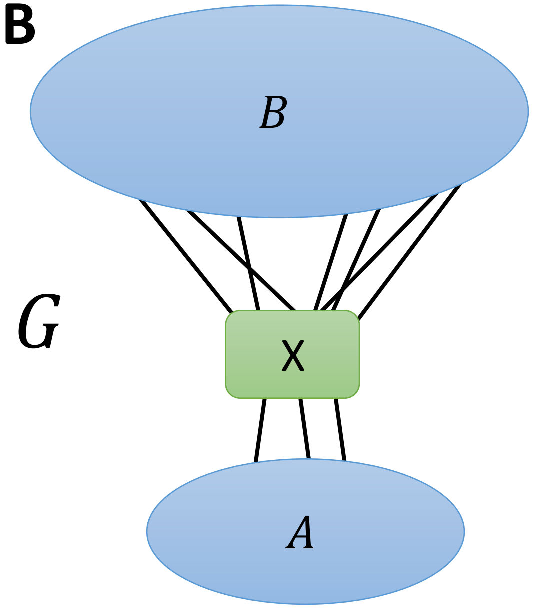
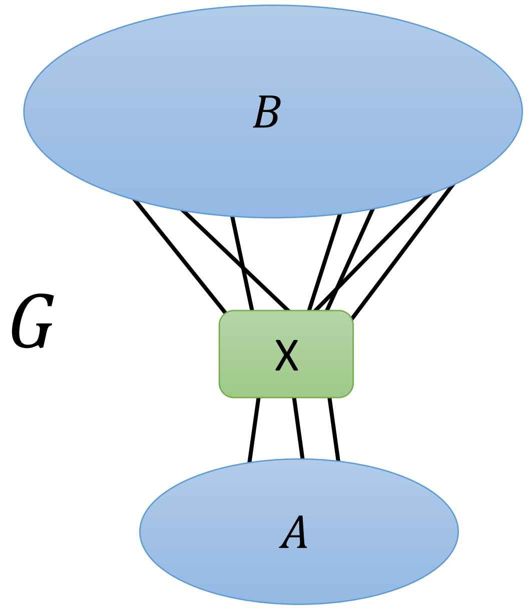


Finding a separator

What is a good separator? Separate the graph into X, A and B, such that

- $|X| \leq \ell + 1$
  - Use a bag in the tree decomposition!
- $|A|$ is small
  - $|A|$ will determine the size of the kernel
  - $|A| = O\left(\frac{\ell^2}{\varepsilon}\right)$
- The part of an optimal solution in $G[A]$ is sufficiently large
  - By discarding X, we loose out on value at most $|X|$
  - $|X|$ should be small, compared to IS(G[A])



Size of A

Theorem

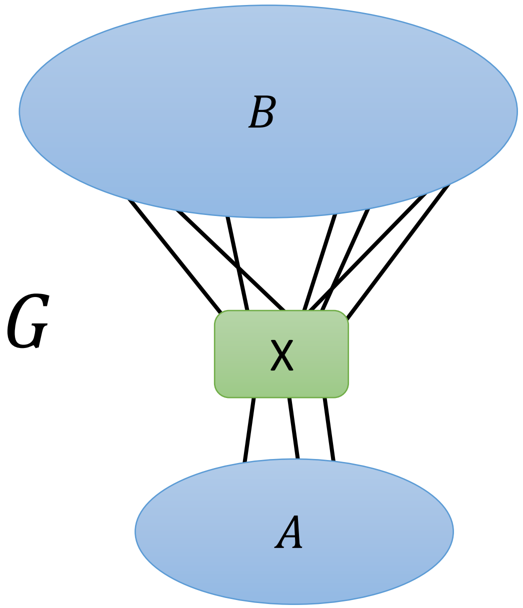
A graph with n vertices and treewidth ℓ, has an independent set of size at least $\frac{n}{\ell+1}$

Proof

Various options, immediate from alternative definition of TW

Conclusion

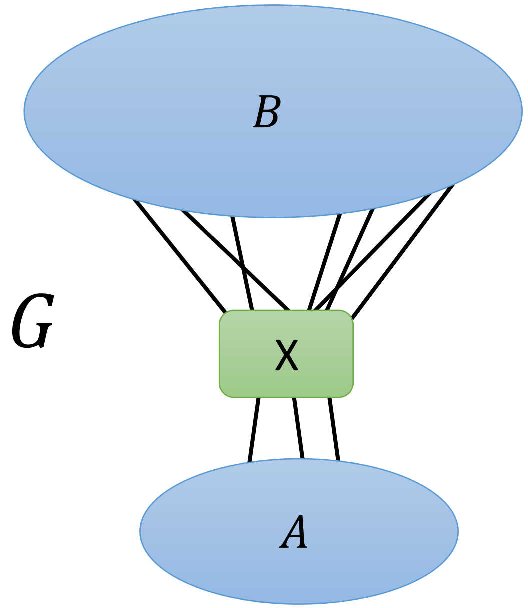
If $|A| \geq \frac{(\ell+1)^2}{\varepsilon}$, then $IS(A) \geq \frac{\ell+1}{\varepsilon} \geq \frac{|X|}{\varepsilon}$



Finding a separator

Find a node t in T such that $\frac{(\ell+1)^2}{\varepsilon} \leq |G_t - \mathrm{X}_t| \leq \frac{10(\ell+1)^2}{\varepsilon}$

- Let $A := G_t - X_t$, $X := X_t$
- Recurse as long as $G_t - \mathrm{X}_t$ too large
  - **Join node** – Recurse on subtree with at least half the vertices
  - **Introduce/forget node** – Recurse on subtree
  - **Leaf node** – Contradicts $G_t - \mathrm{X}_t$ large



Various separator theorems for treewidth are known; we show this one for completeness.

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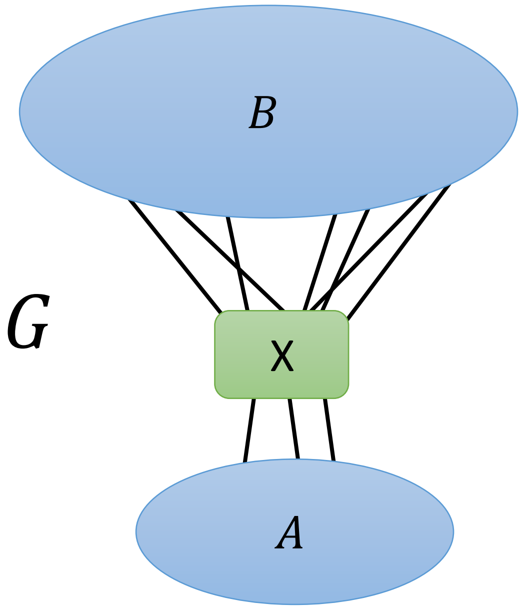

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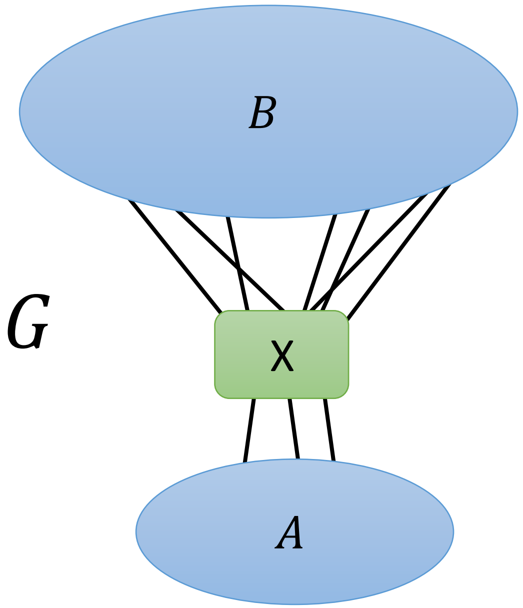


Independent Set: Correctness

Consider an optimal solution S, then

$$|S| = |S \cap A| + |S \cap B| + |S \cap \mathrm{X}| \leq opt(G[A]) + opt(G[B]) + |X|$$

$$\leq c|S_A| + c(1+\varepsilon)|S_B| + \varepsilon|S_A|$$

$$\leq c(1+\varepsilon)(S_A + S_B)$$

Crucial point: Lower bound for IS on graphs of low treewidth

Independent Set: Correctness

Consider an optimal solution S, then

$$|S| = |S \cap A| + |S \cap \ldots X| \leq opt(G[A]) + opt(G[B]) + |X|$$

By the oracle

Induction

$$\leq c|S_A| + c(1+\varepsilon)|S_B| + \varepsilon|S_A|$$

$$\leq c(1+\varepsilon)(S_A + S_B)$$

Crucial point: Lower bound for IS on graphs of low treewidth

Approximate Turing kernel for Vertex Cover

Parameterized by treewidth

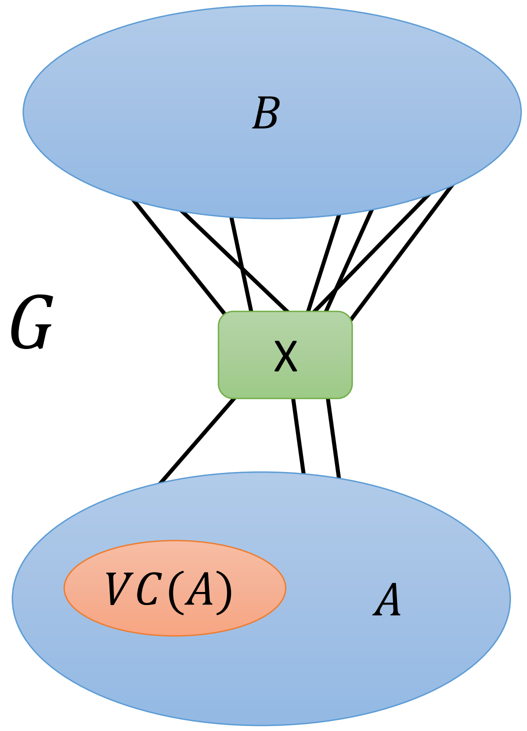
Vertex Cover

Theorem

Vertex Cover has a $(1+\varepsilon)$-approximate Turing Kernel with $O\left(\frac{\ell}{\varepsilon}\right)$ vertices.

Overview

1. Find a good separator X, separate the graph into A and B
   - Such that $VC(G[A])$ small
2. Apply the kernel for vertex cover to $G[A]$
3. Ask the oracle for a solution $S_A{}'$ of A'
4. Use this to obtain a solution S_A of $G[A]$
5. Recurse to find an approximate solution S_B for part B
6. Show $S_A \cup S_B \cup X$ is a $(1+\varepsilon)$-approximate solution

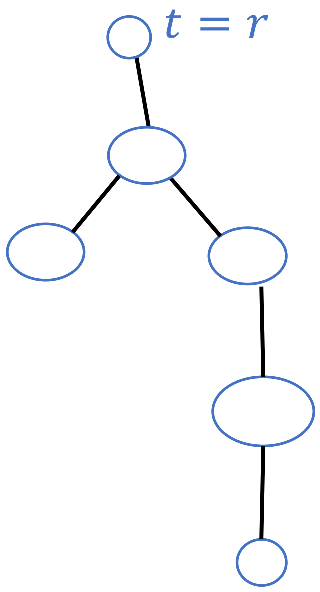


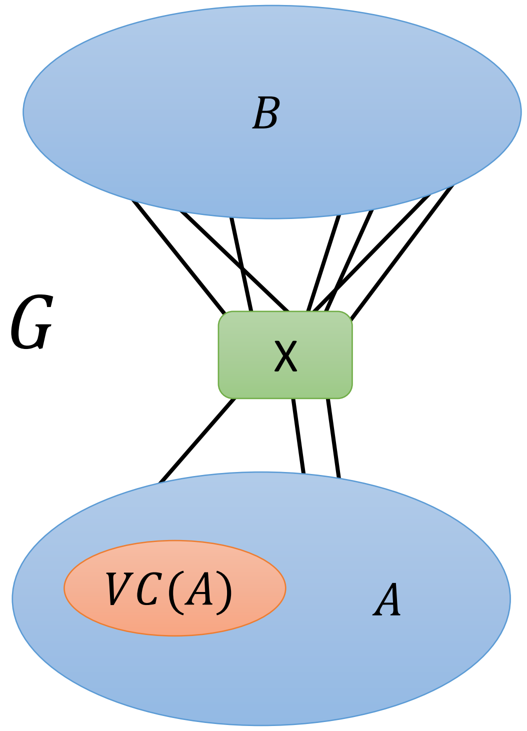
Finding a separator

Find a node t such that

$$\frac{\ell+1}{\varepsilon} \leq VC(G_t - \mathrm{X}_t) \leq \frac{10(\ell+1)}{\varepsilon}$$

Start from the root r





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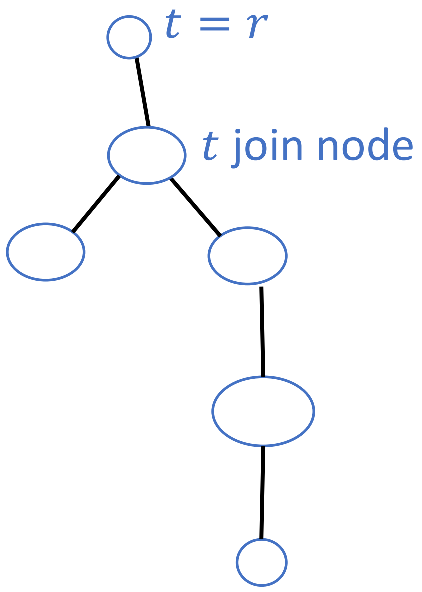

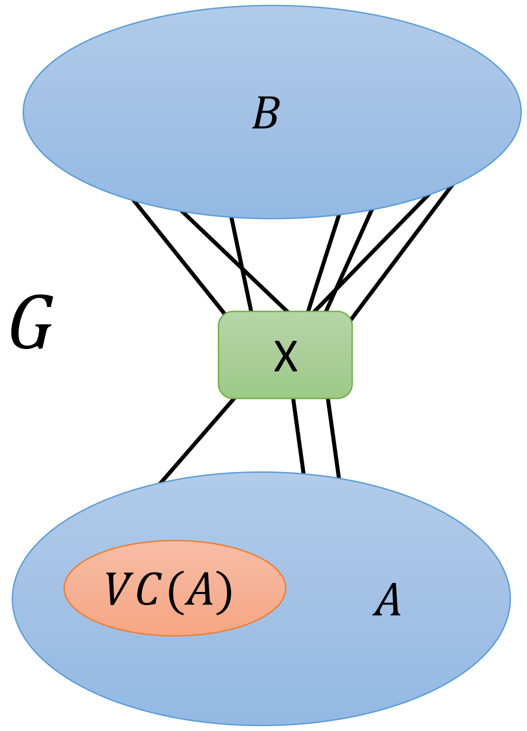

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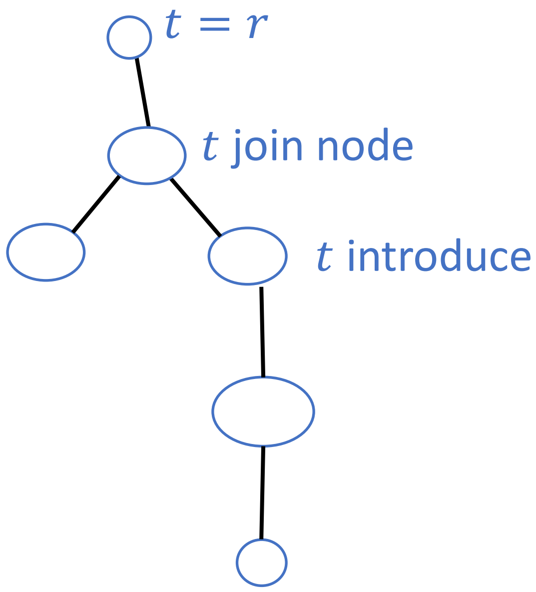

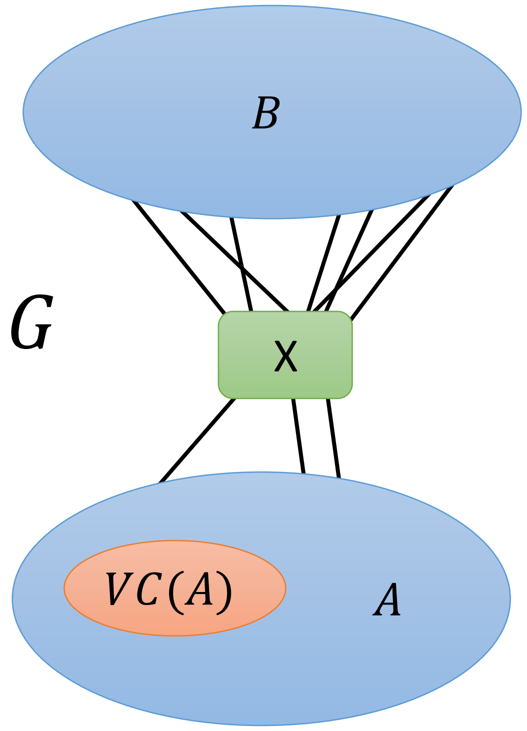

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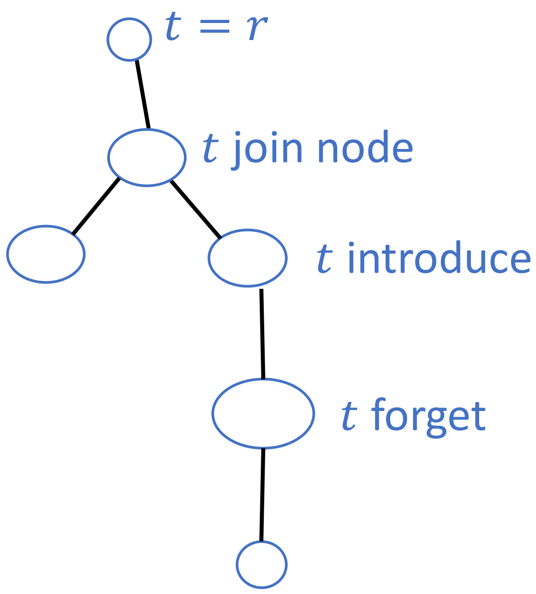

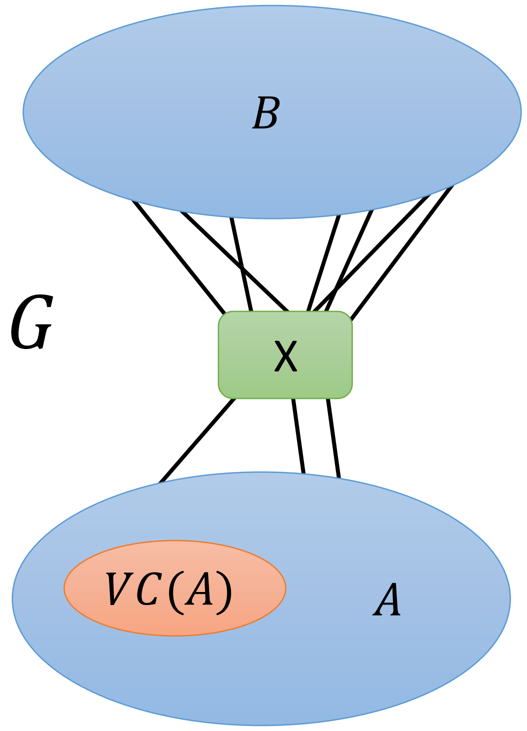

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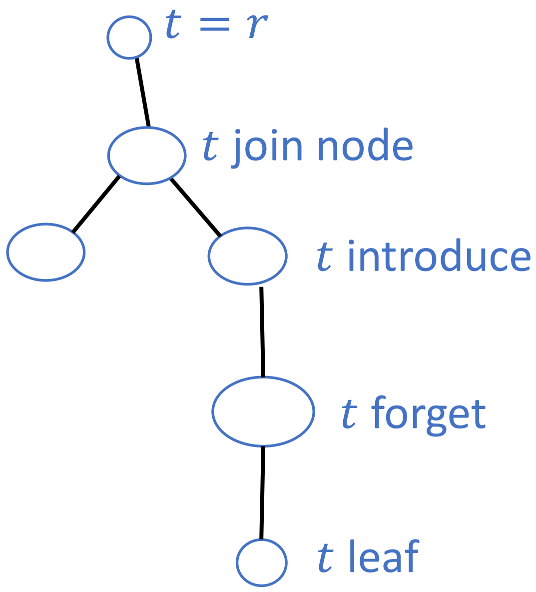

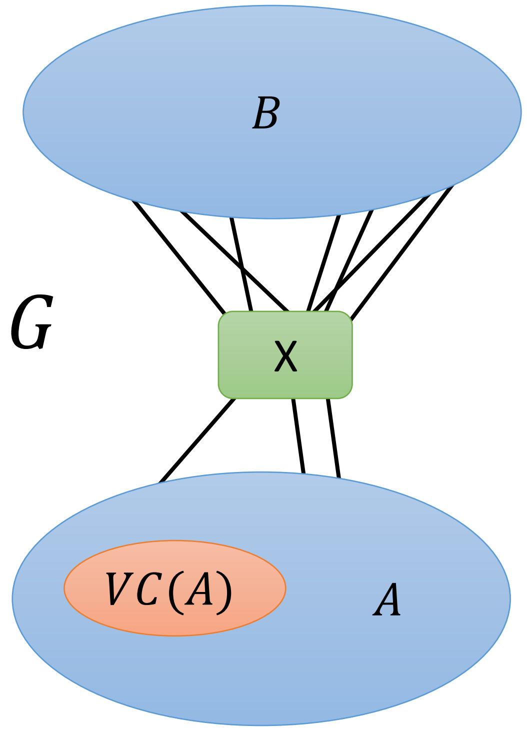

Finding a separator

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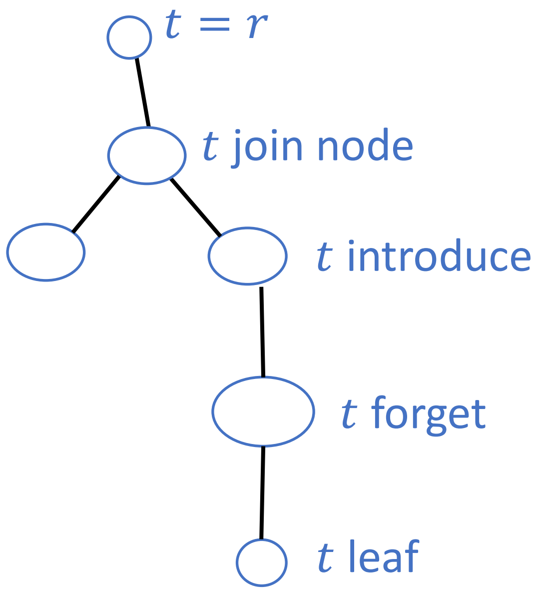
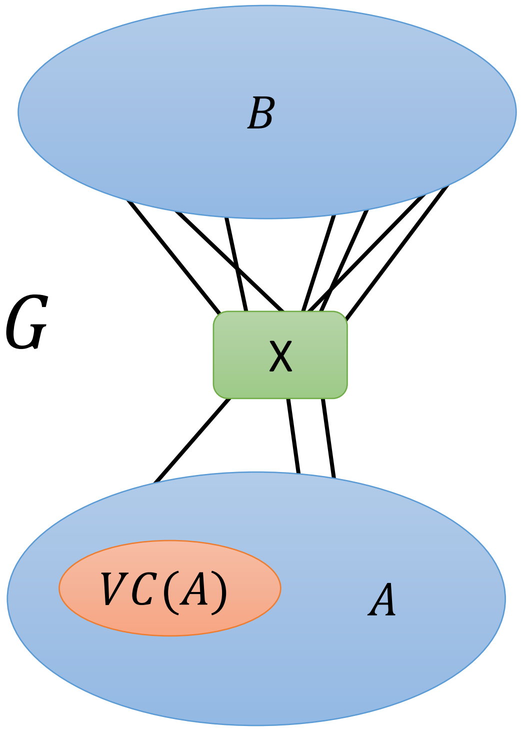


One issue:
Computing Vertex Cover is NP-hard, so how to find t?

Solution
Approximate!



NO Turing kernel for

Vertex Cover

Parameterized by treewidth

Turing kernel lower bound

Theorem

Vertex Cover parameterized by treewidth is MK[2]-hard

If Q is MK[2]-hard, then a poly Turing kernel for Q implies a poly Turing kernel for CNF-SAT(n)

- Believed to not exist

Lower bound proof

Reduction from CNF-SAT

$$F = (x_1 \lor x_2 \lor \cdots \lor x_m) \land (\neg x_3 \lor x_5) \land (x_1 \lor \neg x_2 \lor x_4 \lor x_8) \land \cdots$$

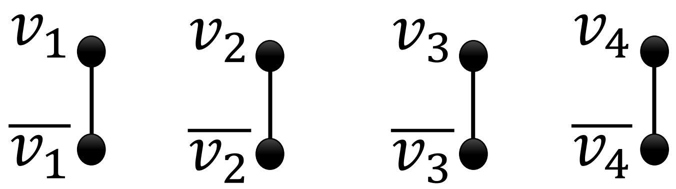
Unbounded clause length

Turing kernel lower bound

$$F = \underbrace{(x_1 \vee x_2)}^{C_1} \wedge \underbrace{(\neg x_1 \vee x_3 \vee x_4)}^{C_2} \wedge \cdots$$

Turing kernel lower bound

$$F = \overset{C_1}{(x_1 \vee x_2)} \wedge \overset{C_2}{(\neg x_1 \vee x_3 \vee x_4)} \wedge \cdots$$



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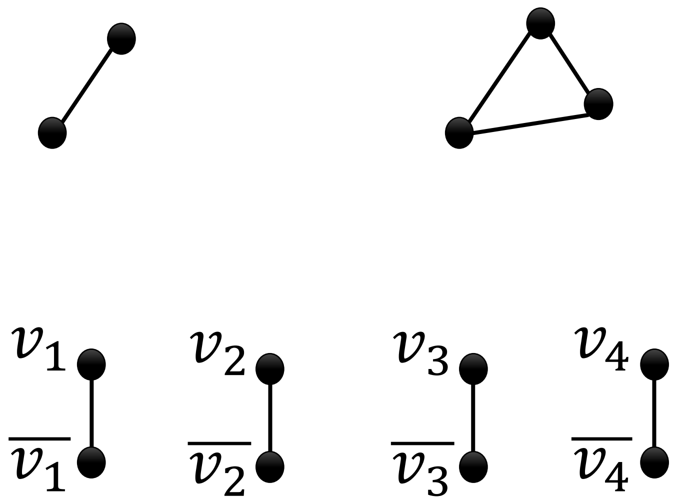
$$F = \overset{C_1}{(x_1 \lor x_2)} \land \overset{C_2}{(\neg x_1 \lor x_3 \lor x_4)} \land \cdots$$



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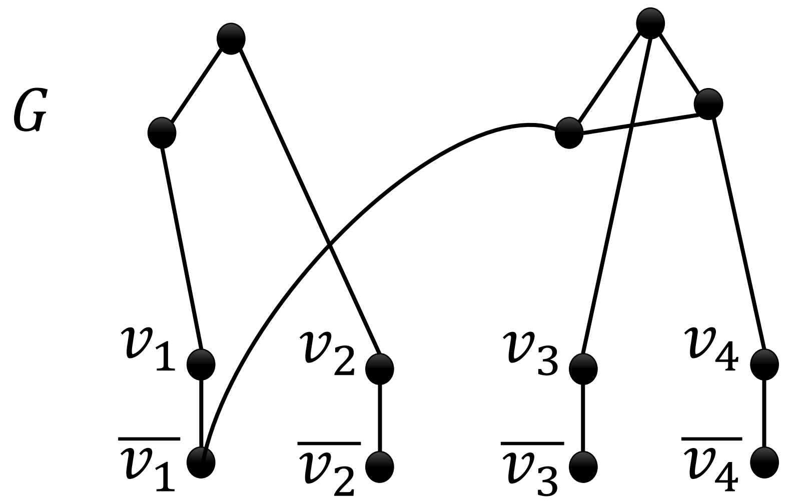

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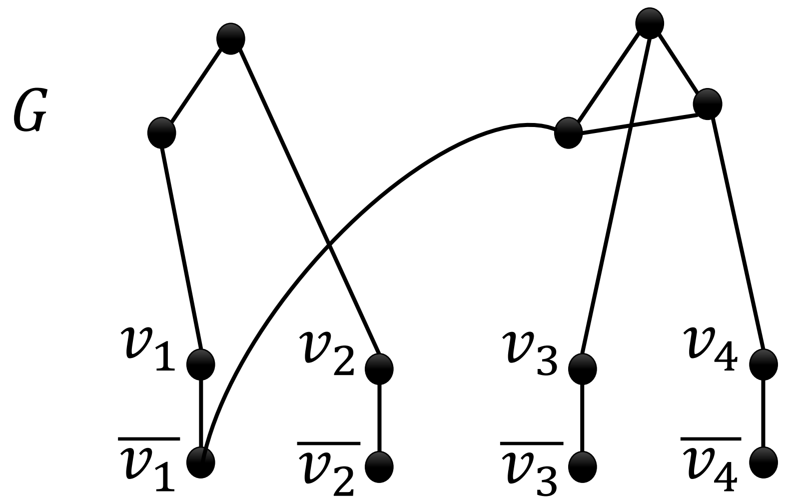
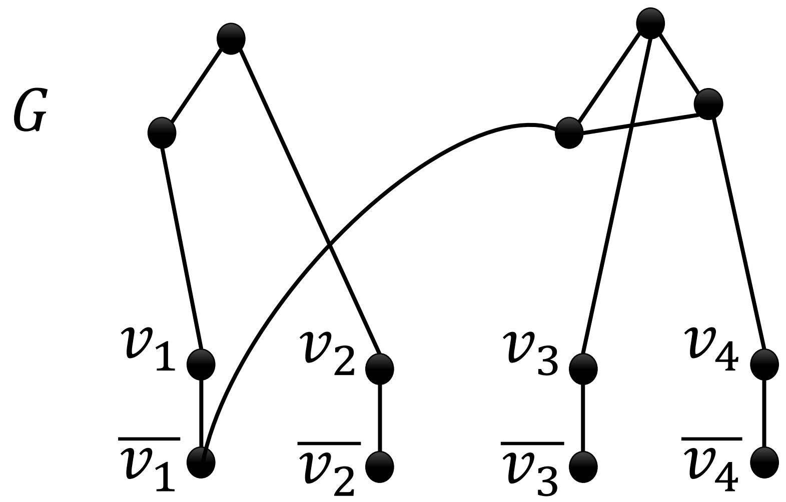


G has a vertex cover of size $n + \sum(|C_i| - 1)$ if and only if F is satisfiable

Turing kernel lower bound

$$\overset{C_1}{F = (x_1 \lor x_2)} \land \overset{C_2}{(\neg x_1 \lor x_3 \lor x_4)} \land \cdots$$

G has treewidth $O(n)$



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Turing kernel lower bound

$$\overset{C_1}{} \quad \overset{C_2}{}$$

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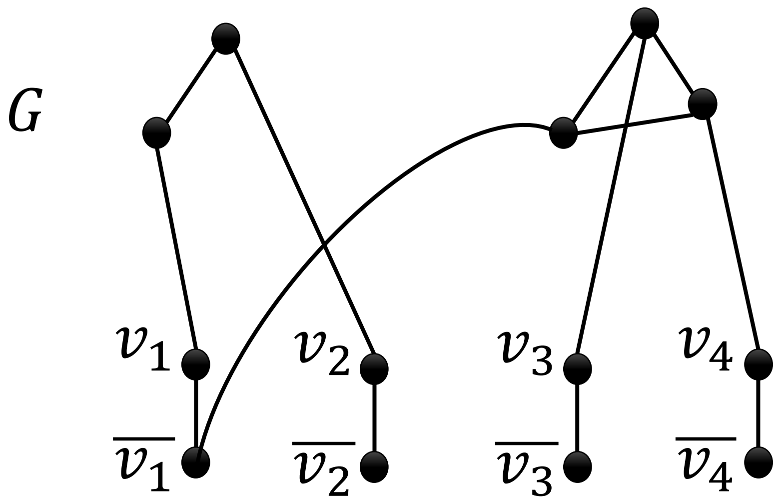

G has a vertex cover of size $n + \sum(|C_i| - 1)$ if and only if F is satisfiable

$\Rightarrow$ If VC(tw) has a polynomial (Turing) kernel, then so does CNF-SAT(n)

Approximate Turing kernel for

Connected Vertex Cover

Parameterized by treewidth

Connected Vertex Cover

Given a graph G (and tree decomposition T) find minimum vertex cover S such that $G[S]$ is connected

Cannot apply earlier idea immediately

- No lower bound based on treewidth
- No polynomial kernel with parameter k
- Combining solutions is complex
  - Need to ensure connectivity

Connected Vertex Cover

No polynomial kernel parameterized by solution size, but

- A $(1+\delta)$-approximate kernel for all $\delta > 0$
  [Lokshtanov, Panolan, Ramanujan, Saurabh STOC 2017]

No good bounds on optimal solution depending on $CVC(G[A])$, $CVC(G[B])$, and X

- Recall for vertex cover we implicitly used

$$\mathrm{VC}(\mathrm{G}[\mathrm{A}]) + VC(G[B]) \leq VC(G) \leq VC(G[A]) + VC(G[B]) + |X|$$

Connected Vertex Cover

No polynomial kernel parameterized by solution size, but

- A $(1 + \delta)$-approximate kernel for all $\delta > 0$
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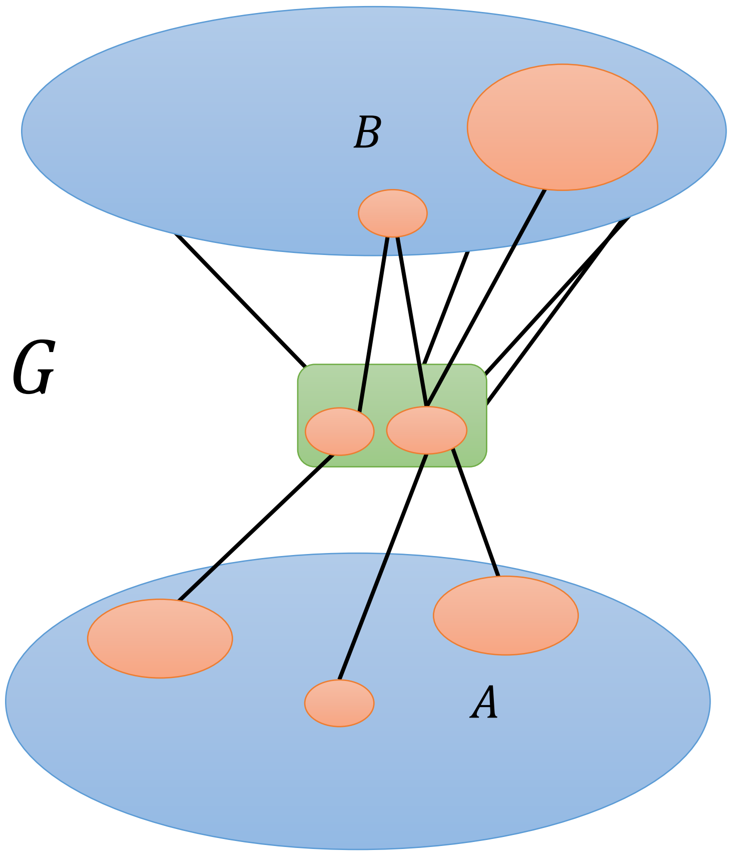
$$\mathrm{VC}(\mathrm{G}[\mathrm{A}]) + VC(G[B]) \leq VC(G) \leq VC(G[A]) + VC(G[B]) + |X|$$

False for CVC, even when $G[A]$ connected

Also problematic

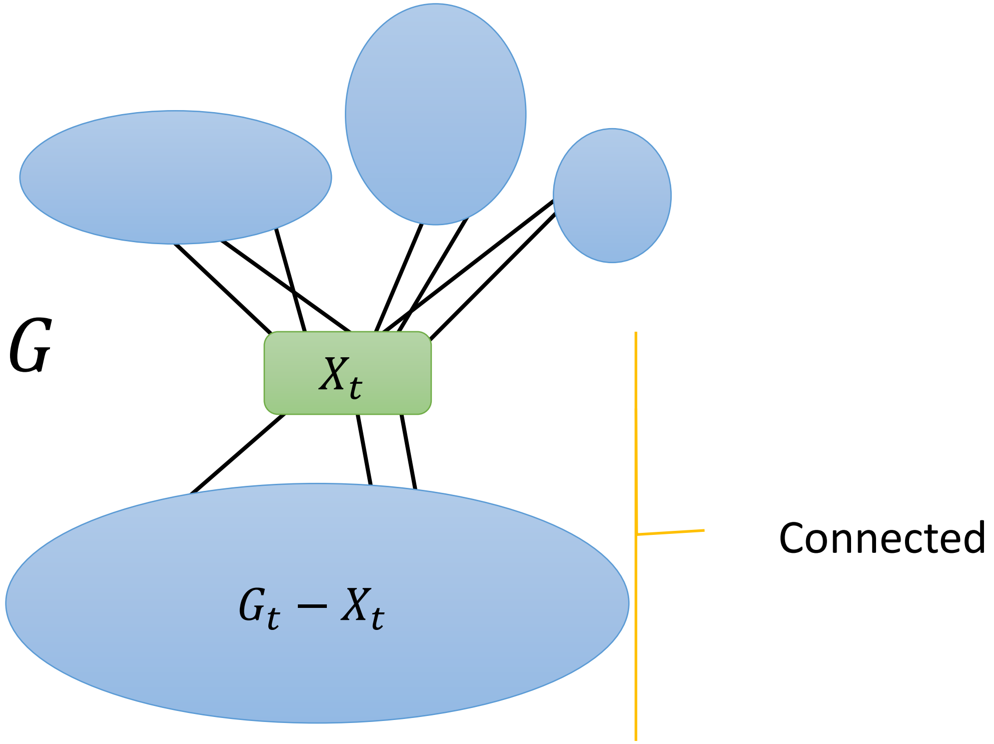


Subconnected tree decompositions

Tree decomposition such that G_t is connected for all t

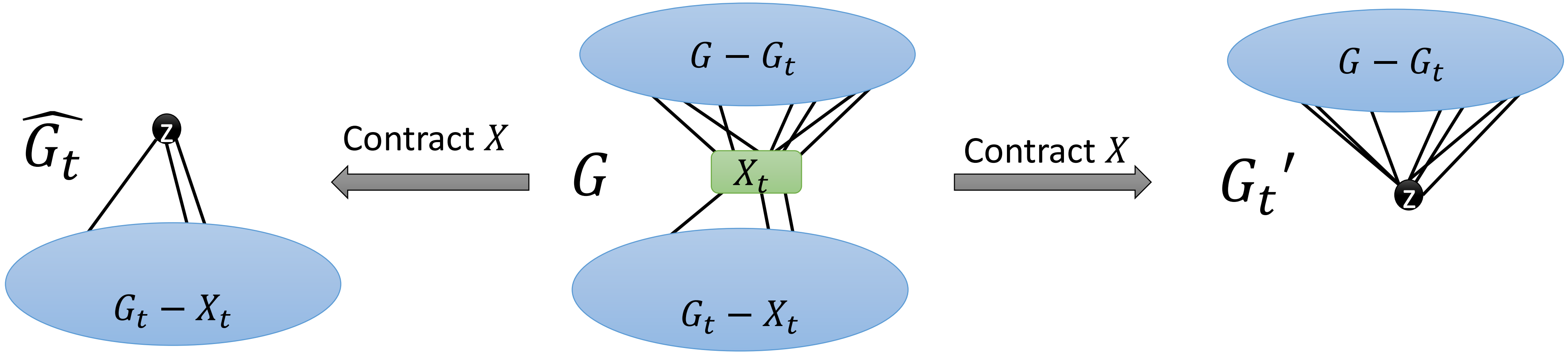
- A given tree decomposition can be made subconnected in polynomial time
  - Without increasing its width

    [Fraigniaud, Nisse, LATIN 2006]



Connected Vertex Cover

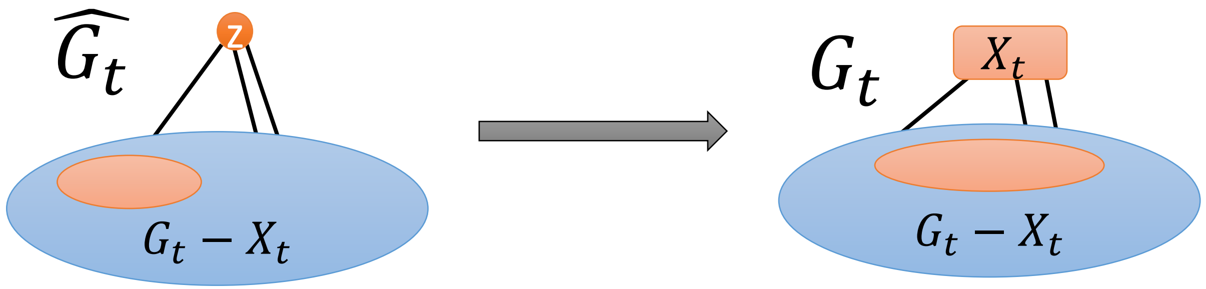
1. If our graph has a small CVC
   - Apply $(1+\varepsilon)$-approximate kernel, obtain (G', k')
   - Feed (G', k') to oracle, obtain solution S'
   - Lift S' to a solution S of (G, k)
2. Else, obtain tree decomposition such that G_t connected for all t
   - For all t, define the following graphs



Connected Vertex Cover

Lemma

Given a connected vertex cover S in $\widehat{G_t}$, we can in polynomial time find a connected vertex cover S' in G_t such that $|S'| \leq |S| + 2|X|$. Furthermore, $X \subseteq S'$.



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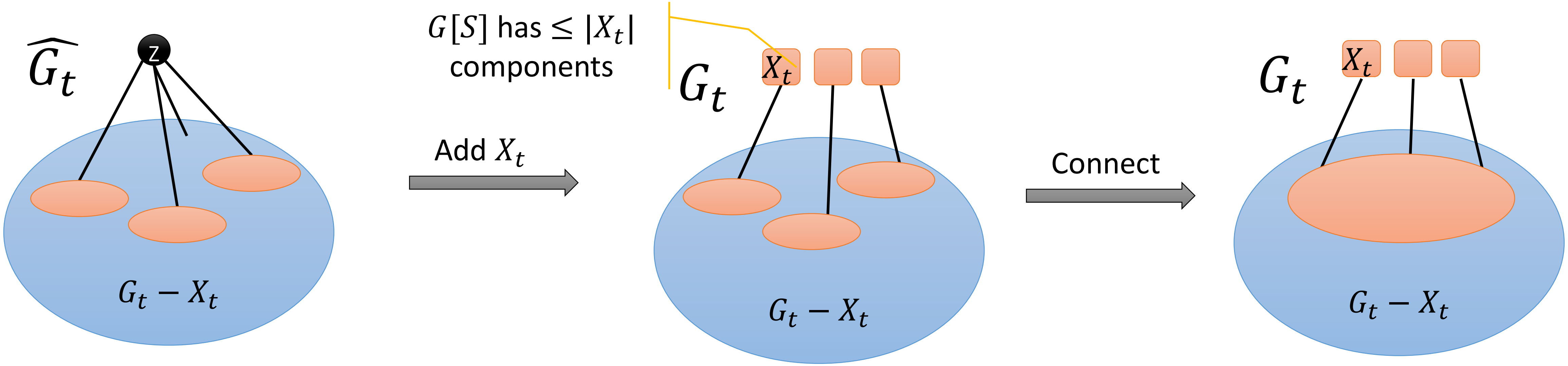

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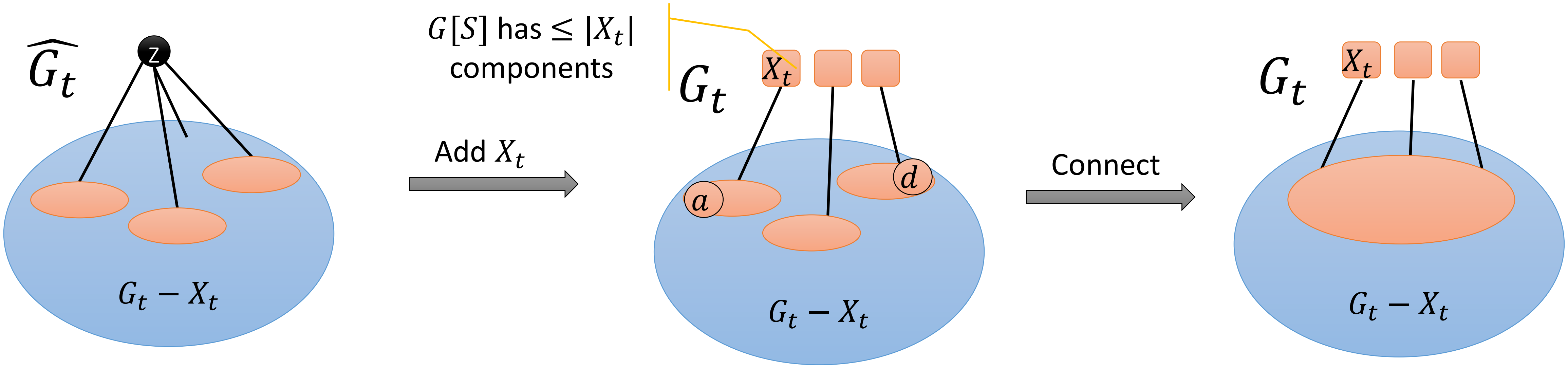

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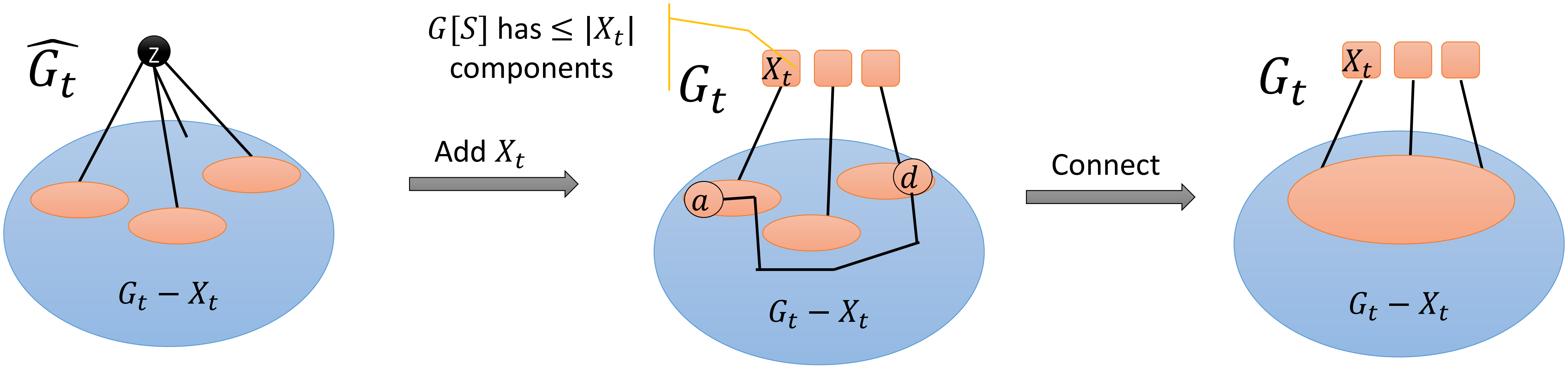

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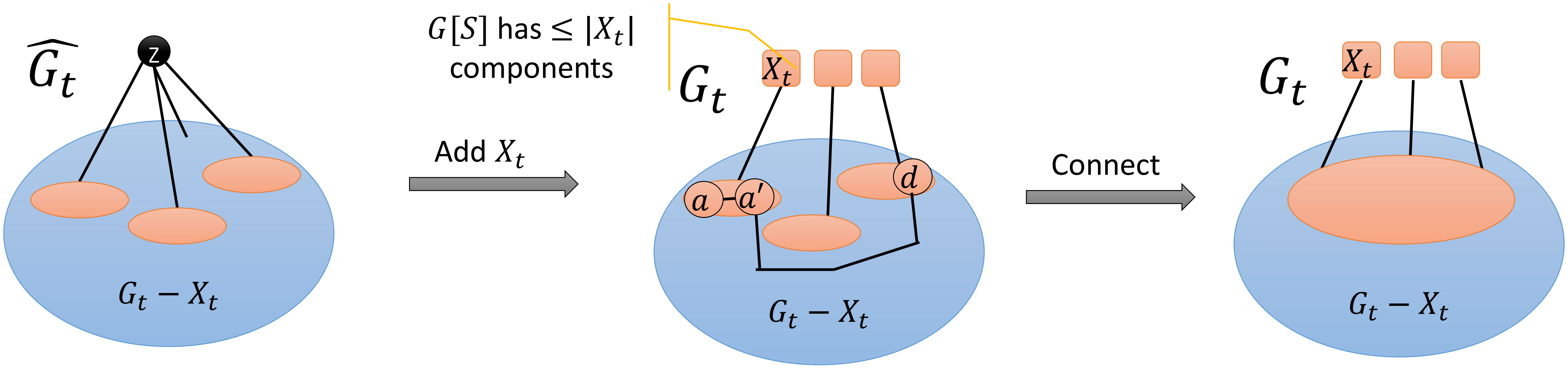

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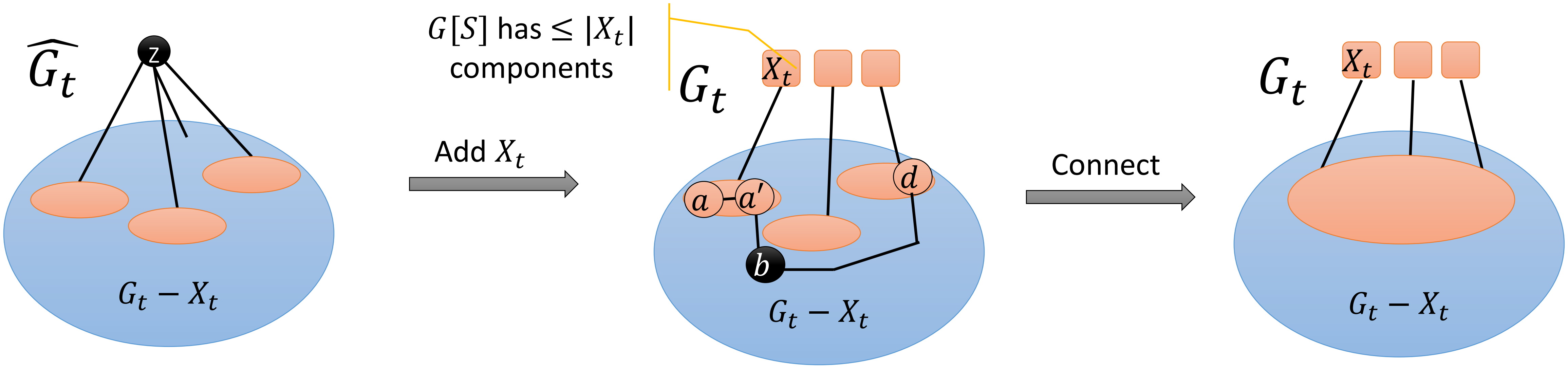

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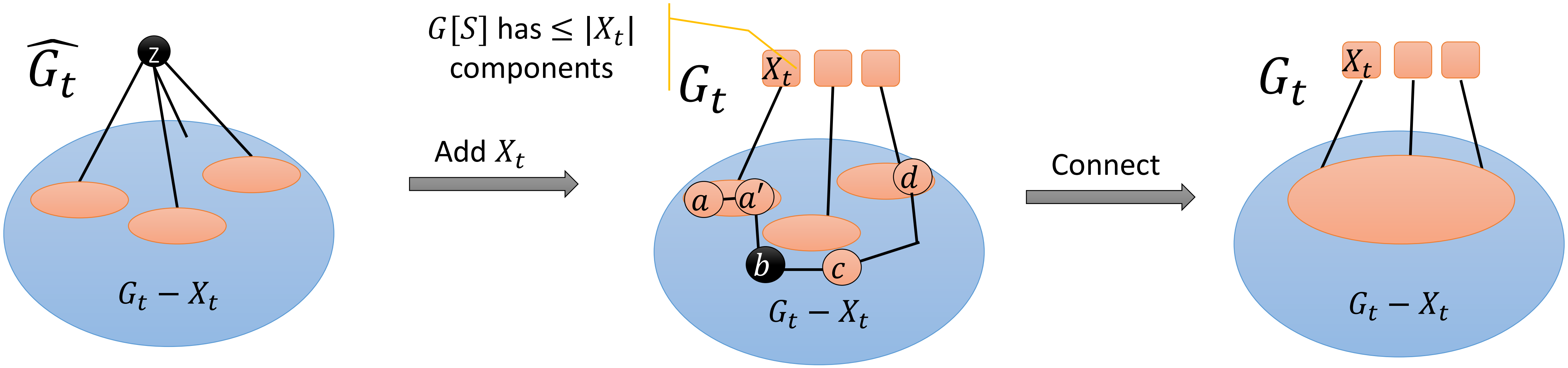

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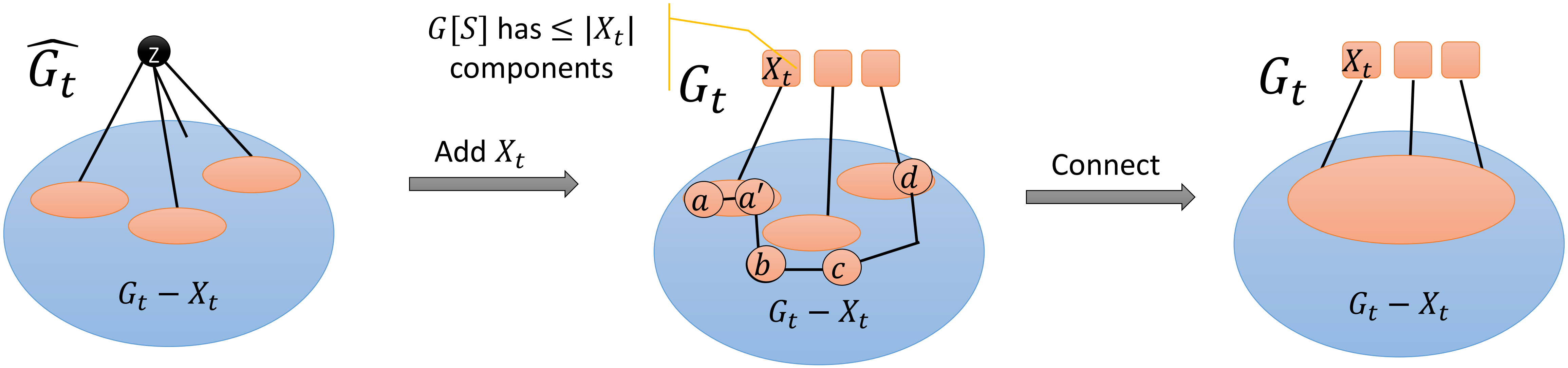


Connected Vertex Cover

1. If G has small CVC
   - Use the $(1+\varepsilon)$-approximate kernel & oracle to obtain $c(1+\varepsilon)$-approx. solution
2. Otherwise, find t such that $\widehat{G_t}$ has CVC of size between $\frac{\ell}{\delta}$ and $\frac{100\ell^2}{\delta}$ for $\delta = \frac{\varepsilon}{3}$
3. Obtain $c(1+\delta)$-approximate CVC $\hat{S}$ in $\widehat{G_t}$
   - Use the $(1+\delta)$-approximate kernel & oracle
4. By lemma, obtain CVC $\tilde{S}$ in G_t, with $X \subseteq \tilde{S}$ and $|\tilde{S}| \leq |\hat{S}| + 2|X|$
5. Obtain $c(1+\varepsilon)$-approximate CVC S' in G'_t
6. Output $S' \cup \tilde{S} \setminus \{z\}$

Approximate Turing kernel for
Friendly Problems

Parameterized by treewidth

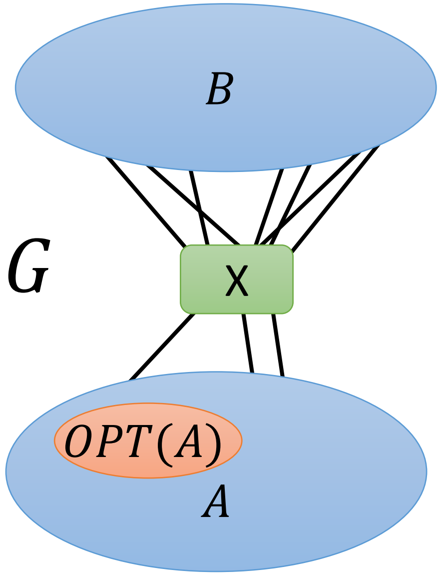
Ingredients for our ATK

A friendly problem

1. Has poly-size $(1+\varepsilon)$-Approximate kernel when parameterized by solution size

2. Has constant-factor approximation algorithm

3. Has very good behavior with respect to separators
   - Construct "good" solution for G based on solutions for A, B



Ingredients for our ATK

A friendly problem

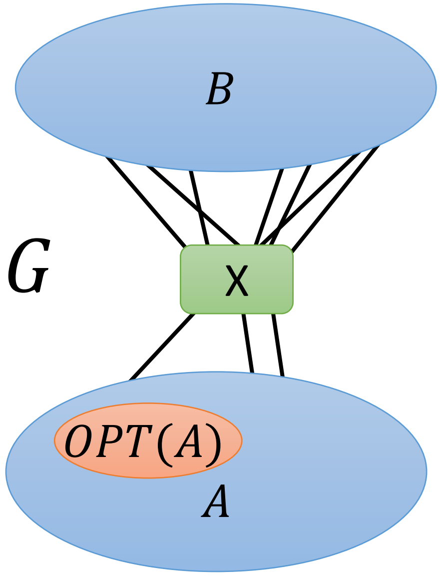
1. Has poly-size $(1+\varepsilon)$-Approximate kernel when parameterized by solution size

Can be relaxed to $k+\ell$

2. Has constant-factor approximation algorithm

Can be relaxed to function of $k+\ell$

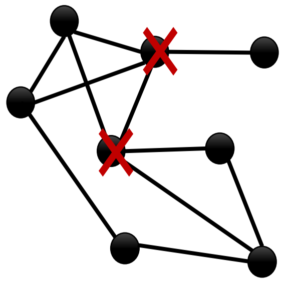
3. Has very good behavior with respect to separators
   - Construct "good" solution for G based on solutions for A, B



Example: Feedback vertex set

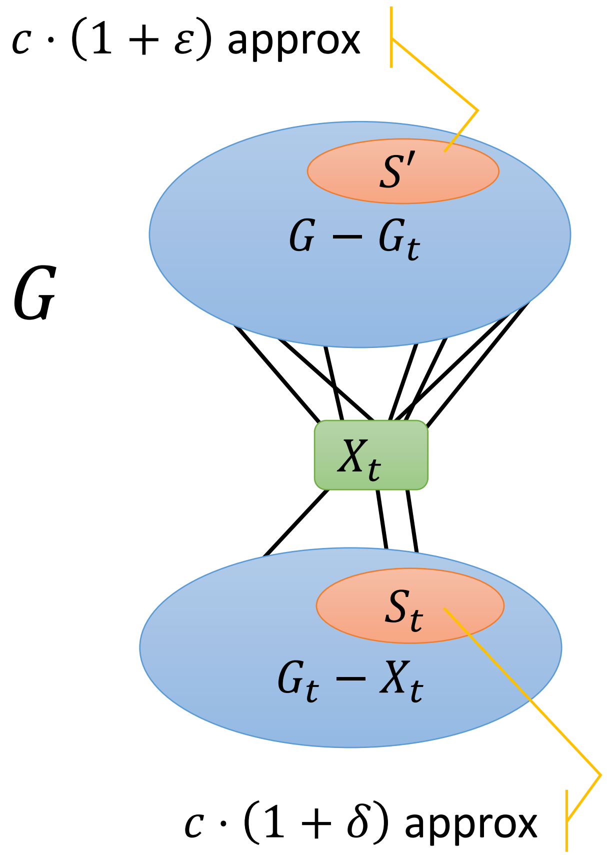
#Vertices we need to remove to make a graph acyclic

- Has kernel parameterized by k
  - Verify: 1-approximate
- Has 2-approximation
- Is otherwise well-behaved
  - If S is a FVS in $G - X$, then $S \cup X$ is a FVS in G



A general strategy (minimization)

1. If our graph has a small optimal solution
   - Apply $(1+\varepsilon)$-approximate kernel, obtain (G', k')
   - Feed (G', k') to oracle, obtain solution S'
   - Lift S' to a solution S of (G, k)
2. Else, find t such that $G_t - \mathrm{X_t}$ has solution size $\geq \frac{g(\ell+1)}{\delta}$
   - Use approximation algorithm
3. Obtain a $c(1+\delta)$-approximate solution S_t for $G_t - X_t$
   - See point 1, use $\delta < \varepsilon$
4. Recurse to obtain $c(1+\varepsilon)$-approximate solution S' of $G - G_t$
5. Combine S_t, S', and info about X_t to a solution in G



Approximate Turing kernels

Conclusions and future work

Summary

Problem	#Vertices in kernel
Independent Set	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex Cover	$O\left(\frac{\ell}{\varepsilon}\right)$
Connected Vertex Cover	$O\left(\left(\frac{\ell^2}{\varepsilon}\right)^{\left\lceil\frac{3+\varepsilon}{\varepsilon}\right\rceil}\right)$
Edge Clique Cover	$O\left(\frac{\ell^4}{\varepsilon}\right)$
Edge-Disjoint Triangle Packing	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex-Disjoint H-Packing for connected H	$O\left(\left(\frac{\ell}{\varepsilon}\right)^{\lvert V(H)\rvert-1}\right)$
Clique Cover	$O\left(\frac{\ell^4}{\varepsilon^2}\right)$
Feedback vertex Set	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$
Edge Dominating Set	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$

These problems parameterized by treewidth ℓ have $(1+\varepsilon)$-approximate Turing Kernels

- Assuming tree decomposition on input
- For all $0 < \varepsilon \leq 1$

Friendly problems have a $(1+\varepsilon)$-approximate Turing kernel with

$$h\left(\frac{\varepsilon}{3}, \varphi\left(\frac{6 \cdot \mathrm{g}(\ell+1)}{\varepsilon} + g(1), \ell\right) + \ell\right)$$

vertices

Summary

Problem	#Vertices in kernel
Independent Set	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex Cover	$O\left(\frac{\ell}{\varepsilon}\right)$
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Size of $(1+\varepsilon)$-approx. kernel

…blems have a $(1+\varepsilon)$-approximate …g kernel with

$$h\left(\frac{\varepsilon}{3}, \varphi\left(\frac{6 \cdot \mathrm{g}(\ell+1)}{\varepsilon} + g(1), \ell\right) + \ell\right)$$

vertices

Summary

Problem	**#Vertices in kernel**
INDEPENDENT SET	$O\left(\frac{\ell^2}{\varepsilon}\right)$
VERTEX COVER	$O\left(\frac{\ell}{\varepsilon}\right)$
CONNECTED VERTEX COVER	$O\left(\left(\frac{\ell^2}{\varepsilon}\right)^{\left\lceil\frac{3+\varepsilon}{\varepsilon}\right\rceil}\right)$
EDGE CLIQUE COVER	$O\left(\frac{\ell^4}{\varepsilon}\right)$
EDGE-DISJOINT TRIANGLE PACKING	$O\left(\frac{\ell^2}{\varepsilon}\right)$
VERTEX-DISJOINT H-PACKING FOR CONNECTED H	$O\left(\left(\frac{\ell}{\varepsilon}\right)^{\lvert V(H)\rvert-1}\right)$
CLIQUE COVER	$O\left(\frac{\ell^4}{\varepsilon^2}\right)$
FEEDBACK VERTEX SET	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$
EDGE DOMINATING SET	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$

These problems parameterized by treewidth ℓ have $(1+\varepsilon)$-approximate Turing Kernels

- Assuming tree decomposition on input
- For all $0 < \varepsilon \leq 1$

Size of $(1+\varepsilon)$-approx. kernel

…blems have a $(1+\varepsilon)$-approximate …g kernel with

$$h\left(\frac{\varepsilon}{3}, \varphi\left(\frac{6 \cdot g(\ell+1)}{\varepsilon} + g(1), \ell\right) + \ell\right)$$

Approximation factor of approximation algorithm

Summary

Problem	#Vertices in kernel
Independent Set	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex Cover	$O\left(\frac{\ell}{\varepsilon}\right)$
Connected Vertex Cover	$O\left(\left(\frac{\ell^2}{\varepsilon}\right)^{\left\lceil\frac{3+\varepsilon}{\varepsilon}\right\rceil}\right)$
Edge Clique Cover	$O\left(\frac{\ell^4}{\varepsilon}\right)$
Edge-Disjoint Triangle Packing	$O\left(\frac{\ell^2}{\varepsilon}\right)$
Vertex-Disjoint H-Packing for connected H	$O\left(\left(\frac{\ell}{\varepsilon}\right)^{\lvert V(H)\rvert-1}\right)$
Clique Cover	$O\left(\frac{\ell^4}{\varepsilon^2}\right)$
Feedback vertex Set	$O\left(\frac{\ell^2}{\varepsilon^2}\right)$
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- Assuming tree decomposition on input
- For all $0 < \varepsilon \leq 1$

…blem… …ximate …g kernel with

$$h\left(\frac{\varepsilon}{3}, \varphi\left(\frac{6\cdot g(\ell+1)}{\varepsilon} + g(1), \ell\right) + \ell\right)$$

Size of $(1+\varepsilon)$-approx. kernel

“Friendlyness” (usually $\ell+1$)

Approximation factor of approximation algorithm

Open questions

Approximate Turing kernels for other problems

- Many graph problems are not “friendly”
  - Constant-factor approximate Turing kernel for DOMINATING SET parameterized by treewidth ?
- Extend to other parameters
  - Other width parameters

More lower bounds

- Problems without $(1+\varepsilon)$-approximate (Turing) kernels

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Thank you!