Sparsification Upper and Lower Bounds for Graph Problems and Not-All-Equal SAT

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Abstract

Many problems we care for are known to be NP-complete. Therefore, we do not think they can be solved in polynomial time, so only an exponential time algorithm is known. To speed up the search for a solution, we want to start by preprocessing an input instance. You can try to make the input instance smaller in polynomial time, without changing the answer. After doing so, we can run the slow algorithm on a smaller input. This thesis investigated whether there exist such algorithms that reduce the number of edges in graph problems, and the number of clauses for logical formulas. Such an algorithm is called a sparsification.

We will show that for a number of decision problems on graphs, polynomial-time algorithms cannot compress instances of such problems to equivalent instances, of a possibly different problem, whose encoding size is sub-quadratic in the number of vertices. Here we only want to maintain the solution (YES/NO), therefore $\mathsf{P} = \mathsf{NP}$ would imply that any NP-complete problem can be compressed in polynomial time to a single bit, indicating whether it was a YES- or a NO-instance. For example look at the 4-colorability problem, which asks if we can color all vertices in a graph in such a way that the endpoints of any edge in the graph have distinct colors. Assuming that NP is not contained in coNP/poly, we show that instances of 4-COLORING cannot be compressed to a sub-quadratic encoding in polynomial time. We obtain similar results for a number of other graph problems.

Finally we consider NOT-ALL-EQUAL SAT (NAE-SAT). This is a variant of the well-known satisfiability problem, that plays a central role in the theory of NP-completeness. We show that an instance of NAE-SAT with n variables and d literals per clause can not be compressed to an equivalent instance of size $\mathcal{O}(n^{d-1-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. Furthermore we present a generalized kernel that matches this lower bound.

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1 Introduction

Background Preprocessing has proven to be useful in many applications, such as when dealing with large scheduling problems with constraints. In this case we can seek to reduce the number of constraints, by removing redundant ones. Furthermore we can sharpen existing constraints such that we do not look into partial schedules that can never lead to a good final scheduling.

In practice, algorithms that seem unfeasibly slow in theory can work surprisingly well on large inputs, but this behavior can usually not be explained by the normal worst-case running time analysis. In this type of analysis, we only consider the running time depending on the total size of the input. There may however exist many large inputs for which even an NP-hard problem can be solved much quicker than expected. To analyze this behavior, the notion of a parameterized problem was introduced, where apart from the total input size there is a parameter $k \in \mathbb{N}$ that measures the complexity of a given input instance ([13], [15]). We hope that if the parameter is small, the instance should now be relatively easy to solve.We say that a parameterized decision problem is fixed-parameter tractable if there is an algorithm that, given any input x of length $|x|$ with parameter value k, determines the YES/NO answer to the instance and whose running time is bounded by a polynomial in $|x|$, multiplied by an arbitrary (usually exponential) function in the parameter k. Therefore, we can make a distinction between parameterized problems that are FPT, and parameterized problems that are not believed to be FPT.

To classify problems, methods were designed to find FPT algorithms or prove a parameterized problems is likely to not be FPT. For example, the method of iterative compression (for example used in [22]) can be used to find FPT algorithms for minimization problems. The main idea is to use a compression routine, that given the input instance and a (too large) solution tries to find a smaller solution or output that this is not possible. To make a distinction between FPT problems and problems that are not believed to be FPT, a hierarchy of complexity classes was introduced in [11]. The main distinction is between class FPT, containing problems that can be solved efficiently when the parameter is small, and problems that are hard for W[1]. Classes FPT and W[1] are the parameterized analogues of P and NP, respectively. By their definition, we know $\mathsf{FPT} \subseteq \mathsf{W}[1]$. It is widely believed that $\mathsf{FPT} \neq \mathsf{W}[1]$, similar to the statement that $\mathsf{P} \neq \mathsf{NP}$. Under this assumption, problems that are hard for W[1] are not fixed parameter tractable.

It was proven that INDEPENDENT SET parameterized by the solution size is W[1]-hard, while VERTEX COVER with the same parameter is fixed parameter tractable. Note that an independent set in a graph is a subset of the vertices, such that none are connected by an edge. Vertex cover on the other hand asks for a subset of the vertices such that for every edge at least one of its endpoints is in the set. It is thereby not hard to see that a graph has a vertex cover of size k, if and only if it has an independent set of size $n - k$.

It turns out that the framework of parameterized complexity analysis makes it possible to rigorously analyze the power of polynomial-time preprocessing algorithms, using the concept of kernelization. A kernelization is a polynomial-time algorithm that transforms a parameterized input x with parameter k for a decision problem, into a new input x' with parameter k', such that the following two conditions hold: (1) the answer to (x,k) is YES if and only if the answer to (x',k') is YES, and (2) there is a function $f : \mathbb{N} \to \mathbb{N}$ such that the size of x' and the new parameter value k' are both bounded by $f(k)$. The function f is called the *size* of the kernelization: it gives a guarantee on the amount of data reduction that is achieved in terms of the parameter k. [1] A kernel whose size f is bounded by a polynomial, is called a *polynomial kernel*. It is known that every problem in FPT has a (not necessarily polynomial) kernel [2]. To apply these kernels in practice, it is desirable to find kernels that are as small as possible.

Methods were introduced to be able to find kernels, or prove that a problem does not have a kernel of a certain size. First, the focus was on deciding whether certain problems had polynomial kernels [3]. Under certain assumptions we can prove a kernel of polynomial size does not exist by giving a cross-composition. This is a polynomial time algorithm that transforms a large number of inputs into a single output of the goal problem that acts as a logical OR of the given inputs. With the right bounds on the parameter value of the output instance, kernel lower bounds can be proven.

It is easy to see some assumptions are needed in order to prove that a problem does not have a kernel of small size. Since many of the problems that we consider are NP-complete, $\mathsf{P} = \mathsf{NP}$ would imply every problem in NP has a trivial polynomial kernel. The preprocessing procedure can in this case decide whether it is given a YES- or a NO instance. Depending on the result, it can then output a trivial, constant size, YES/NO-instance of the same problem. To actually achieve kernelization lower bounds we currently need an even stronger assumption than $\mathsf{P} \neq \mathsf{NP}$, which is $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$. As the reader may know, a problem is in coNP if it can be solved by a co-nondeterministic Turing machine. coNP/poly is a complexity class containing some more problems, namely the ones that can be solved be a co-nondeterministic Turing machine that is allowed to use a polynomial size advice. This polynomial advice is an additional input string, given on an extra tape, that may only depend on the input length n. Its length should be polynomial in n. There are multiple reasons why we believe $\mathsf{NP} \not\subseteq \mathsf{coNP/poly}$. Intuitively, coNP and NP are somehow incomparable. For problems in NP we can give a certificate to validate a YES-instance, such as a satisfying truth assignment for a Boolean formula. For problems in coNP on the other hand, validating NO-instances is easy with the right certificate. For example, if we ask if a certain Boolean formula is a tautology, a truth assignment for which the formula is false would be a certificate for a NO-instance. It seems like allowing a polynomial amount of advice is not enough to change the situation.

[1] As is customary in the literature, we will sometimes abbreviate kernelization by kernel.

The framework for proving lower bounds can be further refined ([17], [5]), to be able to also prove bounds on the size of polynomial kernels, such as proving a kernel of bitsize $\mathcal{O}(n^{d-\varepsilon})$ is unlikely to exist for $\varepsilon > 0$. For example, it was proven that VERTEX COVER, parameterized by the solution size k, does not have a kernel of size $\mathcal{O}(k^{2-\varepsilon})$, unless NP $\subseteq$ coNP/poly [8].

Most instances for classical graph problems can trivially be stored in $\mathcal{O}(n^2)$ bits by using an adjacency matrix, when the input is a graph on n vertices. One may wonder if it is possible to reduce the number of edges in the graph in polynomial time, without changing the answer, such that it can be stored in a sub-quadratic number of bits. This process is often referred to as the sparsification of a graph. Let us for example consider the vertex cover problem again. Suppose we want to find a vertex cover of size at most k, and the graph contains a vertex of degree strictly larger than k. If we do not choose this vertex, we would be forced to pick all its neighbors. However, that would exceed the size limit of our vertex cover. As such, we could pick the vertex in the cover, delete the vertex from the graph and decrease k by one. The original graph has a vertex cover of size k if and only if the smaller graph has a vertex cover of size $k-1$. In this way, we have removed at least $k+1$ edges and one vertex. We would like to know whether a combination of such clever reductions can lead to a significant decrease of the size of our instance.

The goal of this thesis is to analyze whether sparsification is possible for a number of classic NP-complete problems in graph theory and logic. We can use the concept of kernelization to answer this question, by choosing the number of vertices to be the parameter. The question whether an instance can be compressed to an equivalent one of sub-quadratic size, therefore turns into the question of whether the parameterization by the number of vertices n in the graph has a kernel with bitsize $\mathcal{O}(n^{2-\varepsilon})$ for some $\varepsilon > 0$. In this way, we obtain lower bounds stating that no kernel of size at most $\mathcal{O}(n^{2-\varepsilon})$ exists, unless NP $\subseteq$ coNP/poly, where n is the number of vertices in the graph. For logic problems we will use the number of variables as the parameter to obtain sparsification lower bounds.

Results We prove that for several graph-theoretical problems, there is no preprocessing algorithm that transforms an n-vertex instance into an equivalent sparse instance, i.e., one with $\mathcal{O}(n^{2-\varepsilon})$ edges. This is established by considering a more general type of preprocessing algorithm called a *generalized kernel* (see Section 2, Definition 2.2), and proving that even this more general type of reduction algorithm cannot exist unless NP $\subseteq$ coNP/poly. Furthermore we study a variant of satisfiability which is NOT-ALL-EQUAL SAT and give a sparsification lower bound and a generalized kernel that matches the lower bound.

First of all we consider FEEDBACK ARC SET. Given a directed graph, this problem asks if it is possible to find k arcs, whose removal results in an acyclic graph. In the next section we use a linear parameter transformation from VERTEX COVER to obtain a

sparsification lower bound for the problem, which shows that there is no polynomial time algorithm that reduces an instance with n vertices to an equivalent one of bitsize $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

In Section 4 we consider the 4-COLORING problem. The problem asks if it is possible to color the vertices of a graph with 4 colors, such that the endpoints of any edge receive distinct colors. We prove that, under the same complexity assumption, no polynomial-time reduction to an equivalent instance of sub-quadratic size is possible. The proof is related to a construction by Jansen and Kratsch, who proved that an instance of the related Chromatic Number problem (can the given graph be properly colored with q colors?) that is large, but whose graph is simple because it has a small vertex cover, cannot be reduced to an equivalent instance whose size is polynomial in that of the vertex cover [18]. For our sparsification lower bound, we use some of the ideas of Jansen and Kratsch, but we introduce several new gadgets.

We then consider LIST COLORING, in list coloring every vertex has a list of allowed colors and we want to find a proper coloring such that each vertex gets a color from its list. In general, LIST COLORING is a more general problem than 4-COLORING, which is just a special case where each list contains colors {1,2,3,4}. By the previous result, a sub-quadratic sparsification for LIST COLORING is unlikely to exist. Therefore, we consider what happens when restricting the input graphs to be planar. Since the number of edges in a planar graph is always linear in the number of vertices [9], it may not seem surprising that we obtain a kernel of sub-quadratic size. However, since every vertex contains a (arbitrarily long) list of possible colors, we need to bound the size of these lists in our kernel.

To obtain a sparsification lower bound for HAMILTONIAN CYCLE, which asks if there is a cycle in the graph that visits every vertex exactly once, we first consider the directed version of this problem. In Section 6 we use DIRECTED HAMILTONIAN CYCLE, which makes it easier to obtain a cross-composition, where we start from HAMILTONIAN PATH on a restricted graph class. The kernelization lower bound for HAMILTONIAN CYCLE is now obtained by showing how to transform an instance of DIRECTED HAMILTONIAN CYCLE to one of HAMILTONIAN CYCLE, in such a way that the lower bound carries over.

In Section 7 we obtain a lower bound for DOMINATING SET. An input to the problem consists of a graph G and integer k. The problem asks if there exists a subset of the vertices S of size at most k, such that every vertex in $V(G) \setminus S$ has a neighbor in S. We use a combination of gadgets that were used to obtain lower bounds for a different parameterization of DOMINATING SET [10]. The same cross-composition can be used to obtain lower bounds for CONNECTED DOMINATING SET, which as the name suggests asks for a dominating set that is connected. We show that these problems do not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. These results carry over to the dual problems, which has interesting implications, since both problems have a kernel with a linear number of vertices, when parameterized by the solution size ([7],

[14]). Our results imply that it is very unlikely that we can further reduce the number of edges in these kernels to a sub-quadratic amount.

Finally, in Section 8, we look at d-NOT-ALL-EQUAL SAT and the less well-known problem d-HYPERGRAPH 2-COLORABILITY. Given a hypergraph where every edge contains at most d vertices, the latter problem asks for a 2-coloring of the vertices such that each edge contains at least one vertex of each color. It is known that NOT-ALL-EQUAL-SAT does not have a generalized kernel of size $\mathcal{O}(n^{d-1-\varepsilon})$, unless NP $\subseteq$ coNP/poly [18], and using a linear parameter reduction we show that this also holds for d-HYPERGRAPH 2-COLORABILITY. Since every clause may contain d literals in d-NAE-Sat, it is not immediately clear why this lower bound is tight. We present a kernel for d-HYPERGRAPH 2-COLORABILITY using $2 \cdot n^{d-1}$ hyperedges, which matches the lower bound. This results in a generalized kernel for d-NAE-SAT. This is surprising when we compare with the "normal" satisfiability problem, since d-CNF-SAT is not believed to have a kernel of size $\mathcal{O}(n^{d-\varepsilon})$ for any $\varepsilon > 0$ [8].

Related work A lot of work on proving kernelization lower bounds with different parameterizations has already been done. FEEDBACK ARC SET is mostly studied when given a tournament as an input, which is a directed graph T such that exactly one of the arcs (u, v) or (v, u) is in T for every pair of vertices u and v. The problem is known to have a kernel whose number of vertices is linear in the parameter k [1]. Note that, since this problem asks for a subset of arcs, k is in this case bounded by the number of edges, not by the number of vertices.

For q-Coloring, which asks if it is possible to properly color the vertices of a graph with at most q colors, a study was done on the size of kernels for a number of different parameterizations [18]. For example the parameterization by the size k of a vertex cover in the graph is considered. It is shown that for q at least four, there is no kernel of size $\mathcal{O}(k^{q-1-\varepsilon})$. Furthermore they obtain a kernel with $\mathcal{O}(k^q)$ vertices that can be encoded in $\mathcal{O}(k^q)$ bits.

LONG PATH and LONG CYCLE asking for a simple path, respectively cycle, of length at least k are also studied under various parameterizations [4]. Parameterized by the size of a vertex cover both problems yield a kernel of quadratic size. When parameterized by the solution size existence of a polynomial kernel would imply NP $\subseteq$ coNP/poly.

The construction of a cross-composition does not always have to be deterministic in order to prove lower bounds. This idea was first used in [20] to prove that a Ramsey-type problem does not have a polynomial kernel, parameterized by k. This problem asks if a given graph contains an independent set or a clique of size at least k. In this thesis, all constructions will be deterministic.

It was already known that d-NAE-SAT does not have a kernel of size $\mathcal{O}(n^{d-1-\varepsilon})$ unless NP $\subseteq$ coNP/poly [18]. By our results it follows that this bound is in some way tight, we can reduce the number of clauses to $\mathcal{O}(n^{d-1})$.

2 Preliminaries

For a (hyper)graph G let $V(G)$ denote its set of vertices and $E(G)$ its set of (hyper)edges. Let a d-hypergraph be a hypergraph where every edge contains at most d vertices. We call a hypergraph *d-uniform* if every edge contains exactly d vertices, by this definition a normal graph is exactly the same as a 2-uniform hypergraph. For a set of vertices $S \subseteq V(G)$ we use $G[S]$ to denote the graph induced by S, more formally $G[S]$ has vertex set S and set of edges $\{\{u,v\} \in S \times S \mid \{u,v\} \in E(G)\}$. Graphs occurring in this thesis are undirected, unless specified otherwise. They do not contain self-loops and there are no duplicate edges. A graph G is *bipartite* if there exists a partitioning of its vertices into sets S and T, such that $G[S]$ and $G[T]$ are edgeless. This means all edges in G have one endpoint in S and one in T.

Let $[r]$ be defined as $[r] := \{x \in \mathbb{N} \mid 1 \leq x \leq r\}$.

We now introduce the framework that will be used to prove our upper and lower bounds.

▶ **Definition 2.1** (Fixed parameter tractable)**.** A *parameterized problem* is a language $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$. This means that such a problem has input instances (x,k), where the second component is called the parameter. We say that such a problem is *Fixed Parameter Tractable (FPT)* if there is a computable function f and an algorithm to determine if $(x,k) \in \mathcal{Q}$ in $\mathcal{O}(f(k) \cdot \text{poly}(|x|))$ time. ◀

There are many possible parameters for a given problem. For example the size of the required solution, or the size of a minimum vertex cover in the graph. Once the parameterization is chosen, upper and lower bounds on kernel sizes (see Definition 2.2) can be derived. Note that in the definition above $f(k)$ can be any computable function, resulting in an unfeasibly slow algorithm for even relatively small values of k.

In this thesis, the number of vertices in the graph will be used as the parameter. It is obvious that the problem will then be FPT and have a polynomial kernel, but this choice allows us to prove sparsification lower bounds using the framework of cross-composition.

Let us start by giving the formal definition of a (generalized) kernel.

▶ **Definition 2.2** ((Generalized) Kernelization [5])**.** Let $\mathcal{Q}, \mathcal{Q}' \subseteq \Sigma^* \times \mathbb{N}$ be parametrized problems and let $h : \mathbb{N} \to \mathbb{N}$ be a computable function. A *generalized kernelization* for $\mathcal{Q}$ into $\mathcal{Q}'$ of size $h(k)$ is an algorithm that on input $(x,k) \in \Sigma^* \times \mathbb{N}$ takes time polynomial in $|x| + k$ and outputs an instance (x',k') such that the following hold:

- The size of x' and the parameter value k' are bounded by $h(k)$
- The instance (x',k') is YES for $\mathcal{Q}'$ is and only if (x,k) is YES for $\mathcal{Q}$

The algorithm is a kernelization, or in short a *kernel* for $\mathcal{Q}$ if $\mathcal{Q} = \mathcal{Q}'$. ◀

It is known that every problem in FPT has a kernel (cf. [2]). Note that for generalized kernels, there are no real restrictions on the output problem $\mathcal{Q}'$. In particular, $\mathcal{Q}'$ does

not have to be decidable. The preprocessing algorithms given in this thesis are actually "useful", they return an instance for the starting problem or some other NP-complete problem. The lower bounds we present even hold against generalized kernels, which is stronger than lower bounds for normal kernels.

To prove kernelization lower bounds, one possibility is to give a AND/OR-cross-composition (Definition 2.4), where we are asked to combine instances of an NP-hard problem into a single instance of the goal problem, that will act as a logical AND or OR on the given inputs. When giving a AND/OR-cross-composition it is often easy to assume that the given input instances are somehow similar. Therefore, the notion of a polynomial equivalence relation is introduced.

► **Definition 2.3** (Polynomial equivalence relation [5])**.** An equivalence relation $\mathcal{R}$ on Σ^* is called a *polynomial equivalence relation* if the following two conditions hold:

- There is an algorithm that given two strings $x, y \in \Sigma^*$ decides whether x and y belong to the same equivalence class in time polynomial in $|x| + |y|$.
- For any finite set $S \subseteq \Sigma^*$ the equivalence relation R partitions the elements of S into a number of classes that is polynomially bounded in the size of the largest element of S. ◄

For graph problems, a polynomial equivalence relation can define input instances to be equivalent if they share the same number of vertices. It can also be useful to require that all instances in the same equivalence class ask for a solution of the same size, such as for a dominating set of size k. For bipartite graphs a polynomial equivalence relation can be used to ensure that the sizes of the two partite sets are the same.

Let us give an example equivalence relation $\mathcal{R}$ that ensures some of these properties and verify that it a polynomial equivalence relation. Suppose our inputs are instances for DOMINATING SET on bipartite graphs. This problems gets a bipartite graph G and integer k as input and asks if there exists a set $S \subseteq V(G)$ of size at most k, such that every vertex in G has a neighbor in S or is contained in S. Given two instances (G, k) and (G', k') where G is bipartite with $V(G) = S \cup T$ and similarly G' is bipartite with $V(G') = S' \cup T'$ and these partitions are given on input. Let these instances be equivalent under $\mathcal{R}$ if and only if $|S'| = |S|$, $|T'| = |T|$ and $k = k'$. It is easy to see that $\mathcal{R}$ is an equivalence relation and that we can decide in polynomial time whether two strings are equivalent under $\mathcal{R}$. It remains to show that the second condition is satisfied, to do this we will first define how the inputs are stored. Inputs are given as the adjacency matrix of the graph, followed by the indices of all vertices in one side of the partition, followed by the parameter k (in unary). Suppose we are given a set S of input strings, and let the largest element have length x, we need to show that the number of equivalence classes in S is polynomial in x. We know that any instance in S has at most x vertices and parameter k bounded by x. As such, the number of different parameter values occurring in S is at most x. The number of combinations of the size of partition sets S and T is bounded by x^2. It follows that the number of equivalence classes under $\mathcal{R}$ is bounded by x^3, which is polynomial as desired.

▶ **Definition 2.4** (AND/OR-cross-composition [5]). Let $L \subset \Sigma^*$ be a language, let $\mathcal{R}$ be a polynomial equivalence relation on Σ^*, and let $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem. An *OR-cross-composition of L into $\mathcal{Q}$ (with respect to $\mathcal{R}$) of cost $f(t)$* is an algorithm that, given t instances $x_1, \ldots, x_t \in \Sigma^*$ of L belonging to the same equivalence class of $\mathcal{R}$, takes time polynomial in $\sum_{i=1}^{t} |x_i|$ and outputs an instance $(y, k) \in \Sigma^* \times \mathbb{N}$ such that the following hold:

CB The parameter k is bounded by $\mathcal{O}(f(t) \cdot (\max_{i=1}^{t} |x_i|^c))$, where c is some constant independent of t.

OR The instance (y, k) is YES for $\mathcal{Q}$ if and only if at least one instance x_i is YES for L.

An AND-*cross-composition of cost $f(t)$* of L into $\mathcal{Q}$ (with respect to $\mathcal{R}$) is an algorithm that instead fulfills properties CB and AND.

AND The instance (y, k) is YES for $\mathcal{Q}$ if and only if all instances x_i are YES for L. ◀

Depending on the cost of a given AND/OR-cross-composition, it is possible to derive lower bounds for the size of generalized kernels of a parameterized problem. We will use AND/OR-cross-compositions of the following cost to obtain our lower bounds.

▶ **Definition 2.5** (Degree-d AND/OR-cross-composition). Let a *degree-d* AND/OR-*cross-composition* be an AND/OR-cross-composition of bounded cost where $f(t) \in \mathcal{O}(t^{1/d+o(1)})$. We will also use *degree-d cross-composition* to refer to the OR variant. ◀

▶ **Theorem 2.6** ([5]). *Let $L \subseteq \Sigma^*$ be a language, let $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem and let d, ε be positive reals. If L has a degree-d* OR*-cross-composition into $\mathcal{Q}$ with cost $f(t) = t^{1/d+o(1)}$, where t denotes the number of instances, and Q has a generalized kernel with size bounded by $\mathcal{O}(k^{d-\varepsilon})$, then $L \in$ coNP/poly. If, additionally, L is NP-hard, then* NP $\subseteq$ coNP/poly. ◀

As such, a degree-2-OR-cross composition can be used to show that a parameterized problem does not have a kernel of subquadratic size, unless NP $\subseteq$ coNP/poly.

Suppose we want to give a degree-2 AND/OR cross-composition to a problem on graphs, such as 4-COLORING. Let us for now use a simple equivalence relation that defines all input graphs with the same number of vertices to be equivalent. Suppose we are given input graphs $G_1, \ldots, G_t$, each consisting of n vertices.

Consider the most basic cross-composition, let the output graph G simply be the union of all input graphs, meaning $V(G) = \bigcup_{i \in [t]} V(G_i)$ and $E(G) = \bigcup_{i \in [t]} E(G_i)$. It is easy to see that we have now satisfied property AND in Definition 2.4, so this is an AND-cross composition. However, it is *not* a degree-2 AND-cross composition, since the constructed graph G has $\sum_{i \in [t]} V(G_i) = n \cdot t$ vertices. The definition of a degree-2 AND-cross-composition only allows us to use $\mathcal{O}(n^c \cdot \sqrt{t})$ vertices for some constant c. Thereby, we can not preserve all vertices of all input instances.

The constructions given in this paper, all rely on the same basic idea that allows us to keep all edges of all input instances, while forgetting about most of the vertices. As a starting problem, choose an appropriate NP-hard problem on a restricted class of graphs.

This graph class is chosen such that the vertices of all input instances can be partitioned into two sets, such that we know the structure of the graph induced by either partite set. For example, a bipartite graph, where each of the sets induces an edgeless graph. Given t instances of your favorite NP-hard problem on bipartite graphs, we construct a graph for our output problem by first creating $2\sqrt{t}$ independent sets of appropriate size. We represent an input instance x by connecting two such independent sets such that the graph they induce is exactly x. It then remains to construct additional gadgets to obtain a logical OR of all inputs. See Figure 1 for an example of this construction using 9 input graphs.

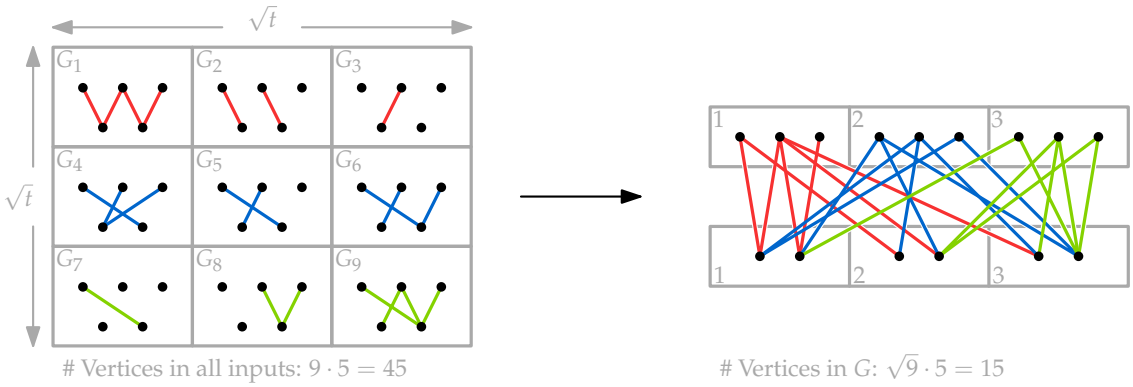


Figure 1. The basic idea used to keep all edges in a degree-2 cross-composition, demonstrated with nine small bipartite input graphs.

In order to use obtained lower bounds to prove lower bounds for other parameterized problems, a linear parameter transformation is defined as follows.

► **Definition 2.7** (Linear parameter transformation [17])**.** Let L_1 and L_2 be two parameterized problems. We say that L_1 is *linear parameter reducible* into L_2, written $L_1 \leq_{\text{lpt}} L_2$, if there exists a polynomial time computable function f mapping instances of L_1 into instances of L_2, such that for all $(x,k) \in \Sigma^* \times \mathbb{N}$, if $(x',k') = f(x,k)$ then:

- $(x,k) \in L_1 \Leftrightarrow (x',k') \in L_2$ and
- $k' = \mathcal{O}(k)$. ◄

In some cases an existing NP-hardness reduction can be used as a linear parameter transformation. We give an example of this in Section 3, where we use the NP-completeness proof of FEEDBACK ARC SET by Karp [19] to obtain a sparsification lower bound for this problem.

► **Theorem 2.8** ([17],[18])**.** *Let L_1 and L_2 be two parameterized problems, and let $d \in \mathbb{N}$ be some constant. If $L_1 \leq_{lpt} L_2$ and L_2 has a generalized kernel of size $\mathcal{O}(k^d)$, then L_1 also has a generalized kernel of size $\mathcal{O}(k^d)$.* ◄

Theorem 2.8 will be used to show that if parameterized problem L_1 does not have a kernel of size $\mathcal{O}(k^d)$, and $L_1 \leq_{\text{lpt}} L_2$, then neither does L_2.

3 Feedback Arc Set

In this section we will show a kernelization lower bound for FEEDBACK ARC SET (FAS). In the FEEDBACK ARC SET problem we are given a directed graph G and an integer k. We ask whether it is possible to remove at most k arcs from G, such that the result is a directed acyclic graph. To prove a kernelization lower bound, we will use a linear parameter transformation from VERTEX COVER. A vertex cover of a graph is a subset S of its vertices, such that for every edge in the graph at least one of its endpoints is contained in S, we also say that this edge is covered by (some vertex in) S. In the VERTEX COVER problem we ask if there exists a vertex cover in the graph that has size at most k.

Dell *et al.* showed a stronger result of the following Lemma, in Theorem 2 in [8].

▶ **Lemma 3.1.** VERTEX COVER *parameterized by the number of vertices n does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless* NP $\subseteq$ coNP/poly. ◀

The reduction given below is equal to the NP-hardness proof of FAS given by Karp in [19].

▶ **Theorem 3.2.** FEEDBACK ARC SET *parameterized by the number of vertices n does not have a generalized kernel of size $O(n^{2-\varepsilon})$, unless* NP $\subseteq$ coNP/poly.

Proof. Suppose we are given an undirected graph $G = (V, E)$ and ask if it has a vertex cover of size at most k. We construct a directed graph $G' = (V', E')$ for feedback arc set in the following way. For every vertex in v, add vertices v_{in} and v_{out} to V' and arc (v_{in}, v_{out}) to E'. For every edge $\{u, v\}$ in E, we add arcs (u_{out}, v_{in}) and (v_{out}, u_{in}) to E'. See Figure 2 for an example. It remains to show validity of this reduction.

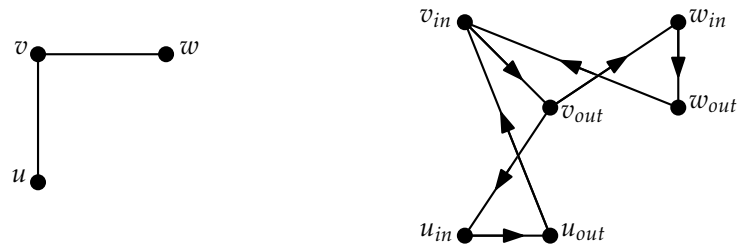


Figure 2. Reduction from VERTEX COVER (left) to FEEDBACK ARC SET (right) on a small graph.

▷ **Claim 3.3.** *G has a vertex cover of size k if and only if G' has a feedback arc set of size k.*

Proof. ($\Rightarrow$) Let G have a vertex cover S of size at at most k. For every vertex $v \in S$, add (v_{in}, v_{out}) to S', which will be the feedback arc set for G'. Therefore it remains to prove that $G'[V' \setminus S']$ is acyclic. Suppose for contradiction that G' contains a cycle after removing all arcs in S' from E'. A cycle cannot consist of only arcs of type (u_{in}, u_{out}), so it must contain an arc (u_{out}, v_{in}) for some $u, v \in V'$. Since this implies that $\{u, v\}$ was an

edge in E, either u or v must be in the vertex cover S of G. If $u \in S$, then (u_{in}, u_{out}) was added to S'. Therefore, vertex u_{out} has in-degree 0 in $G'[V' \setminus S']$ and can never be part of a cycle. Similarly, if $v \in S$, then $(v_{in}, v_{out}) \notin G'[V' \setminus S']$. Consequently, vertex v_{in} has out-degree 0 and can not be part of a cycle. So, either v_{in} or u_{out} can not be contained in a cycle, implying that the edge (u_{out}, v_{in}) is not in a cycle. This contradicts the initial assumption, therefore, $G'[V' \setminus S']$ is acyclic.

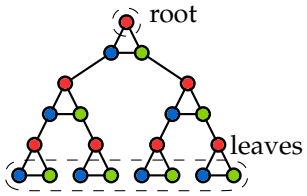
($\Leftarrow$) Now suppose we are given a feedback arc set S' in G' of size at most k. If the FAS contains an arc (u_{out}, v_{in}), we can replace it by the arc (v_{in}, v_{out}) since any cycle traversing the first arc, must also traverse (v_{in}, v_{out}). Therefore, we may assume that is contains only arcs of type (v_{in}, v_{out}). Now we add vertex v to the vertex cover S of G if (v_{in}, v_{out}) is in the feedback arc set of G'. Now suppose for contradiction that S is not a proper vertex cover, so at least one edge $\{u, v\}$ is not covered. This implies that arcs (u_{in}, u_{out}) and (v_{in}, v_{out}) are not present in S'. Since $\{u, v\} \in E$, it follows that arcs (u_{out}, v_{in}) and (v_{out}, u_{in}) are contained in E' and these edges are not present in S' by assumption. But then, $G'[V' \setminus S']$ contains the directed cycle $(u_{in}, u_{out}, v_{in}, v_{out}, u_{in})$, which contradicts that S' is a Feedback Arc Set of G'. So, S must be a Vertex Cover of G. ◁

The reduction given above satisfies all the properties of a linear-parameter transformation from VERTEX COVER parameterized by the number of vertices n, to FEEDBACK ARC SET parameterized by the number of vertices n (Definition 2.7). Suppose FEEDBACK ARC SET does have a sub-quadratic generalized kernel, by Theorem 2.8 this implies that VERTEX COVER also has a sub-quadratic generalized kernel. But then, using Lemma 3.1 it follows that $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. ◀

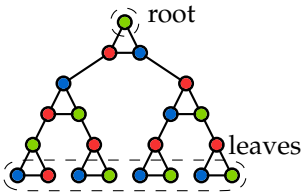
4 4-Coloring

In this section we will look at the 4-COLORING problem. A k-coloring of a graph G is a function $c\colon V(G) \to [k]$. The 4-COLORING problem asks there exists a proper 4-coloring of a given input graph. A graph coloring is proper if the two endpoints of any edge in the graph are assigned distinct colors. We show that 4-COLORING does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$, by giving a degree-2 cross-composition from an NP-hard problem that will be introduced later. Before giving the construction, we first present some of the gadgets that will be needed and their most important properties.

▶ **Definition 4.1** (Treegadget)**.** A *treegadget* is the graph obtained from a complete binary tree by replacing each vertex v by a triangle on vertices r_v, x_v and y_v. Let r_v be connected to the parent of v and let x_v and y_v be connected to the left and right subtree of v. An example of a treegadget with 8 leaves is shown in Figure 3. If vertex v is the root of the tree, then r_v is named the *root* of the treegadget. If v does not have a left subtree, then x_v is a *leaf* of this gadget, similarly, if v does not have a right subtree then we refer to y_v as a leaf of the gadget. Let the *height* of a treegadget be equal to the height of its corresponding binary tree. ◀



(a) Treegadget with no red leaf.



(b) Treegadget where one of the leaves is red.

Figure 3. Treegadgets with example colorings.

It is easy to see that a treegadget is 3-colorable. The important property of this gadget is that if there is a color that does not appear on any leaf in a proper 3-coloring, then this must be the color of the root. Consequently, if the root is not colored with a certain color, then at least one of the leaves must be. See Figure 3a for an illustration. This property will be used to prove that if the graph constructed for the cross-composition has a 4-coloring, at least one input instance was a YES-instance.

▶ **Lemma 4.2.** *Let T be a treegadget with root r and let $c\colon V(T) \to \{1,2,3\}$ be a proper 3-coloring of T. If $k \in \{1,2,3\}$ such that $c(v) \neq k$ for every leaf v of T, then $c(r) = k$.*

Proof. This will be proven using induction on the structure of a treegadget. For a single triangle, the result is obvious. Suppose we are given a treegadget of height h and that the statement holds for all treegadgets of smaller height. Consider the top triangle r, x, y where r is the root. For both subtrees of this triangle, it is clear that all leaves in the

subtree are colored using $\{1,2,3\} \setminus k$. Thus, it follows from the induction hypothesis that the roots of the left and right subtree are colored using k. Hence x and y can not use color k. Since x, y, r is a triangle, r has color k in the 3-coloring. ◀

The following lemma will be used to argue that if one of the input instances is a YES-instance, we can extend the coloring of this input to a proper 4-coloring of the graph constructed for the cross-composition.

▶ **Lemma 4.3.** *Let T be a treegadget with leaves $L \subseteq V(T)$ and root r. Any 3-coloring $c' \colon L \to \{1,2,3\}$ that is proper on $T[L]$ can be extended to a proper 3-coloring of T. If there is a leaf $v \in L$ such that $c'(v) = i$, then such an extension exists with $c(r) \neq i$.*

Proof. We will prove this by induction on the height of the treegadget. For a single triangle, the result is obvious. Suppose the lemma is true for all treegadgets up to height $h-1$ and we are given a treegadget of height h with root triangle r, x, y and with coloring of the leaves c'. Let one of the leaves be colored using i. Without loss of generality assume this leaf is in the left subtree, which is connected to x. By the induction hypothesis, we can extend the coloring restricted to the leaves of the left subtree to a proper 3-coloring of the left subtree such that $c(r_1) \neq i$. We assign color i to x. Since c' restricted to the leaves in the right subtree is a proper 3-coloring of the leaves in the right subtree, by induction we can extend that coloring to a proper 3-coloring of the right subtree. Suppose the root of this subtree gets color $j \in \{1,2,3\}$. We now color y with a color $k \in \{1,2,3\} \setminus \{i,j\}$, which must exist. Finally, choose $c(r) \in \{1,2,3\} \setminus \{i,k\}$. By definition, the vertices r, y, and x are now assigned a different color. Both x and y have a different color than the root of their corresponding subtree, thereby c is a proper coloring. We obtain that the defined coloring c is a proper coloring extending c' with $c(r) = i$. An example is shown in Figure 3b. ◀

▶ **Definition 4.4** (Triangular gadget)**.** A *triangular gadget* is the graph on 12 vertices depicted in Figure 4. Vertices u, v, and w are the *corners* of the gadget, all other vertices are referred to as *inner vertices*. ◀

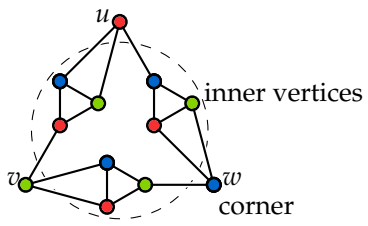


Figure 4. Triangular gadget.

It is easy to see that a triangular gadget is always 3-colorable in such a way that every corner gets a different color. Moreover, we can prove the following lemma.

▶ **Lemma 4.5.** *Let G be a triangular gadget with corners u,v and w and let $c\colon V(G) \to \{1,2,3\}$ be a proper 3-coloring of G. Then $c(v) \neq c(u) \neq c(w) \neq c(v)$. Furthermore, every partial coloring that assigns distinct colors to the three corners of a triangular gadget can be extended to a proper 3-coloring of the entire gadget.*

Proof. Suppose there is a proper 3-coloring c of the entire gadget, such that $c(u) = c(v) = i$ for two distinct corners of the gadget and $i \in \{1,2,3\}$. Note that the inner vertices of the gadget consist of three triangles. There is one such triangle for which every of its vertices has either u or v as a neighbor. This implies that color i cannot be used in this triangle, which means that c can never be a proper 3-coloring of the gadget.

Given a partial coloring that assigns distinct colors to the corners, it is easy to extend this coloring to a coloring of the entire gadget, as shown in Figure 4. ◀

This concludes the description of the gadgets that will be used. We now define the source problem for the cross-composition. It is a modification of 3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION, which is given as input a graph G with partition of its vertex set into $X \cup Y$ such that $G[X]$ is an edgeless graph and $G[Y]$ is a disjoint union of triangles. Such a graph is called a triangle split graph, the problem asks if this graph has a proper 3-coloring. The problem we define furthermore requires that the independent set is colored using only 2 colors, as defined below. 3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION was used by Bodlaender *et al.* to prove kernel lower bounds for CHROMATIC NUMBER parameterized by the size of a vertex cover [5].

2-3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION

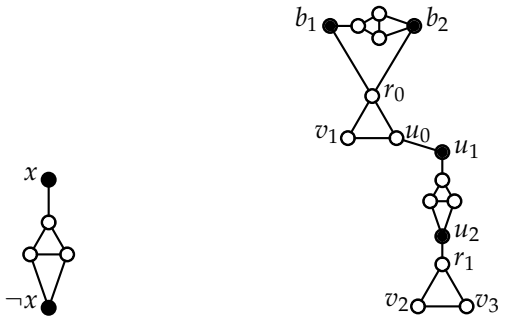
Input: A graph G with a partition of its vertex set into $X \cup Y$ such that $G[X]$ is an edgeless graph and $G[Y]$ is a disjoint union of triangles.

Question: Is there a proper 3-coloring $c : V(G) \to \{1,2,3\}$ of G, such that $c(x) \in \{1,2\}$ for all $x \in X$? We will refer to such a coloring as a *2-3-coloring* of G.

▶ **Lemma 4.6.** 2-3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION *is NP-complete.*

Proof. It is easy to verify the problem is in NP. We will show that it is NP-hard by giving a reduction from 3-NOT-ALL-EQUAL-SATISFIABILITY (3-NAE-SAT), which is known to be NP-complete [16]. For the 3-NAE-SAT problem we are given a Boolean formula in conjunctive normal form (CNF). This means we are given a conjunction of clauses, where each clause is a disjunction of exactly three literals. We ask if there exists a truth assignment to the variables such that every clause contains at least one *true* and at least one *false* literal.

Suppose we are given formula $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ over set of variables U. Construct graph G in the following way. For every variable $x \in U$, construct a variable gadget as depicted in Figure 5a. For every clause C_i, construct a clause gadget as depicted in Figure 5b. For $C_i = (\ell_1 \vee \ell_2 \vee \ell_3)$ with $i \in [m]$, connect vertex ℓ_j for $j \in \{1,2,3\}$ to vertex v_j in gadget C_i in G.



(a) Gadget for a variable **(b)** Gadget for a clause

Figure 5. The gadgets constructed for the clauses and variables of F.

It is easy to verify that G has a triangle split decomposition. In Figure 5, triangles are shown with white vertices and the independent set is shown in black.

Suppose G is 2-3-colorable with color function $c : V(G) \to \{1,2,3\}$ and let $c(v) \in \{1,2\}$ for all v in the independent set. Note that in each of the pairs $\{x, \neg x\}$, $\{b_1, b_2\}$, and $\{u_1, u_2\}$ the two vertices have distinct colors in any proper 2-3-coloring of G. To satisfy F, let $x = \mathit{true}$ if and only if $c(x) = 2$. To show that this results in a satisfying assignment, consider any clause C_i for $i \in [m]$. Note that $c(x) = 2 \Leftrightarrow c(\neg x) = 1$. Since $c(b_1) \neq c(b_2)$ and $c(b_1), c(b_2) \in \{1,2\}$ we obtain $c(r_0) = 3$. Therefore, v_1 and u_0 are colored using colors 1 and 2.

Suppose $c(v_1) = 1$. In the construction we added an edge from v_1 to ℓ_1 and thereby $c(\ell_1) = 2$. This implies that the first literal of C_i is set to *true*. By $c(u_0) = 2$, we know $c(u_1) = 1$ and $c(u_2) = 2$. Thereby, $c(r_1) \neq 2$, so either $c(v_2) = 2$ or $c(v_3) = 2$. If $c(v_2) = 2$, then $c(\ell_2) = 1$ which implies that literal ℓ_2 is *false* in C_i. Similarly, if $c(v_3) = 2$, then $c(\ell_3) = 1$ which implies that literal ℓ_3 is *false* in C_i. In both cases it follows that clause C_i is NAE-satisfied.

When $c(v_1) = 2$, we can use the same argument with the colors 1 and 2 swapped, to show that ℓ_1 is *false* in C_i and ℓ_2 or ℓ_3 is *true*, which again implies that C_i is NAE-satisfied.

Suppose F is a YES-instance, with satisfying truth assignment S. Define color function $c\colon V(G) \to \{1,2,3\}$ as $c(x) := 1$ and $c(\neg x) := 2$ if x is set to *false* in S, define $c(x) := 2$ and $c(\neg x) := 1$ otherwise. Color the remainder of the variable gadgets consistently. We now need to show how to color the clause gadgets. Consider any clause $C_i = (\ell_1 \vee \ell_2 \vee \ell_3)$. At least one of the literals is *true* and one is set to false, by symmetry we only consider four cases. The corresponding colorings are depicted in Figure 6, where red corresponds to 1, green corresponds to 2 and blue corresponds to color 3. It is easy to verify that this leads to a proper 3-coloring that only uses colors 1 and 2 on vertices in the independent set. ◀

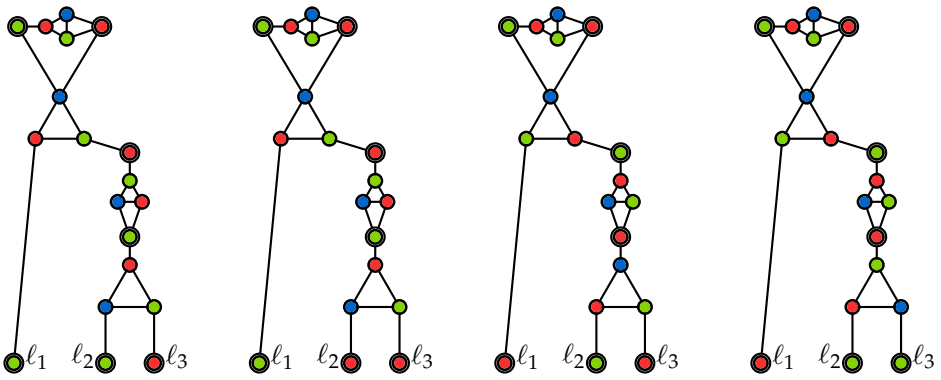


Figure 6. Colorings of a clause gadget, depending on the coloring of the literals $\ell_1, \ldots, \ell_3$. Note that if the roles of ℓ_2 and ℓ_3 are exactly reversed, you can just exchange colors between their parents to get a proper coloring for that situation.

► **Theorem 4.7.** 4-COLORING *parameterized by the number of vertices n does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless* NP $\subseteq$ coNP/poly.

Proof. By Theorem 2.6 and Lemma 4.6 it suffices to give a degree-2 cross-composition from the 2-3-coloring problem defined above into 4-COLORING parameterized by the number of vertices. For ease of presentation, we will actually give a cross-composition into the 4-LIST COLORING problem, whose input consists of a graph G and a list function that assigns every vertex $v \in V(G)$ a list $L(v) \subseteq [4]$ of allowed colors. The question is whether there is a proper coloring of the graph in which every vertex is assigned a color from its list. The 4-LIST COLORING problem reduces to the ordinary 4-COLORING by a simple transformation that adds a 4-clique to enforce the color lists, which will prove the theorem. For now, we focus on giving a cross-composition into 4-LIST COLORING.

We start by defining a polynomial equivalence relation on inputs of 2-3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION. Let two instances of 2-3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION be equivalent under equivalence relation $\mathcal{R}$ when they have the same number of triangles and the independent sets have the same size. It is easy to see that $\mathcal{R}$ is a polynomial equivalence relation. By duplicating one of the inputs, we can ensure that the number of inputs to the cross-composition is an even power of two; this does not change the value of OR, and increases the total input size by at most a factor four. We will therefore assume that the input consists of t instances of 2-3-COLORING WITH TRIANGLE SPLIT DECOMPOSITION such that $t = 2^{2i}$ for some integer i, implying that $\sqrt{t}$ and $\log \sqrt{t}$ are integers. Let $t' := \sqrt{t}$. Enumerate the instances as $X_{i,j}$ for $1 \leq i, j \leq t'$. Each input $X_{i,j}$ consists of a graph $G_{i,j}$ and a partition of its vertex set into sets U and V, such that U is an independent set of size m and $G_{i,j}[V]$ consists of n vertex-disjoint triangles. Enumerate the vertices in U and V as $u_1, \ldots, u_m$ and $v_1, \ldots, v_{3n}$, such that vertices $v_{3\ell-2}, v_{3\ell-1}$ and $v_{3\ell}$ form a triangle, for $\ell \in [n]$. We will create an instance G' of the 4-LIST-COLORING problem, which consists of a graph G' and a list

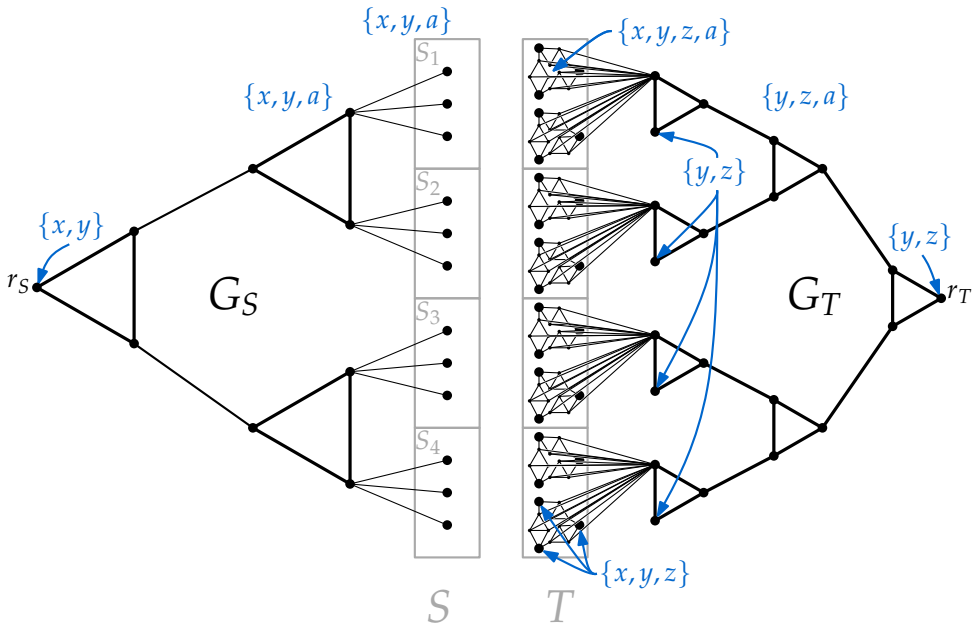


Figure 7. Skeleton of the construction that is used when combining $t = 16$ instances of 2-4-Coloring on Triangle Split Graphs into one instance of 4-List Coloring, implying $t' = \sqrt{16} = 4$. Each input is partitioned into an independent set of size $m = 3$ and a set of $n = 2$ triangles. The lists of allowed colors for different vertices in the graph are visualized by arrows. Edges between vertices in S and T are left out for simplicity.

function L that assigns each vertex a subset of the color palette. For ease of presentation we use the letters $\{x, y, z, a\}$ to indicate the four colors, instead on the numbers one to four. Refer to Figure 7 for a sketch of G'.

We start the construction by creating the basic graph that keeps all input edges as described in the preliminaries, where we get triangle-split graphs instead of bipartite graphs as inputs.

1. Initialize G' as the graph containing t' sets of m vertices each, called S_i for $i \in [t']$. Label the vertices in each of these sets as s^i_ℓ for $i \in [t']$, $\ell \in [m]$ and let $L(s^i_\ell) := \{x, y, a\}$.

2. Add t' sets of n triangular gadgets each, labeled T_j for $j \in [t']$. Label the corner vertices in T_j as t^j_ℓ for $\ell \in [3n]$, such that vertices $t^j_{3\ell-2}, t^j_{3\ell-1}$ and $t^j_{3\ell}$ are the corner vertices of one of the gadgets for $\ell \in [n]$. Let $L(t^j_\ell) := \{x, y, z\}$ and for any inner vertex v of a triangular gadget, let $L(v) := \{x, y, z, a\}$.

3. Connect vertex s^i_k to vertex t^j_ℓ if in graph $G_{i,j}$ vertex u_k is connected to v_ℓ, for $k \in [m]$ and $\ell \in [3n]$. By this construction, the subgraph of G' induced by $S_i \cup T_j$

is isomorphic to the graph obtained from $G_{i,j}$ by replacing each triangle with a triangular gadget.

We will now create the additional structure, that allows us to select one instance that should be 4-colorable. First, we add one treegadget that will ensure that at least one of the groups in S is colored using only the colors x and y.

4. Add a treegadget G_S with t' leaves to G' and enumerate these leaves as $1, \ldots, t'$; recall that t' is a power of two. Connect the i'th leaf of G_S to every vertex in S_i. Let the root of G_S be r_S and define $L(r_S) := \{x, y\}$. For every other vertex v in G_S let $L(v) := \{x, y, a\}$.

For T we create a treegadget that ensures that for one group of T, the entire group (including the inner vertices of gadgets) is colored using $\{x, y, z\}$. In other groups, the idea is to color the inner vertices using $\{x, y, a\}$ and the corner vertices using color z.

5. Add a treegadget G_T with $2t'$ leaves to G' and enumerate these leaves as $1, \ldots, 2t'$. For $j \in [t']$, connect every inner vertex of a triangular gadget in group T_j to leaf number $2j - 1$ of G_T. For every leaf v with an even index let $L(v) := \{y, z\}$ and let the root r_T have list $L(r_T) := \{y, z\}$. For every other vertex v of gadget G_T let $L(v) := \{y, z, a\}$.

By construction, G_T has twice as many leaves as G_S. It is important to notice that using a similar construction as for G_S –having one leaf per group– is not desirable here. A leaf in G_T is connected to all inner vertices of the triangular gadgets in some group. Consider a group in T that will not be part of our solvable input instance, we want to color its inner vertices using $\{x, y, a\}$. This implies the leaf connected to this group is always colored using z. However, we have many such groups and two neighboring leaves cannot both get color z. Therefore, we use twice as many leaves where the odd leaves connect to the groups, while the even leaves are just there to fill the tree and keep the nice property of a treegadget we proved in Lemma 4.2.

We can now verify that the created graph G' acts as a logical OR of the given input instances.

▷ **Claim 4.8.** *The graph G' is 4-list colorable ⇔ some input instance X_{i^*,j^*} is 2-3-colorable.*

Proof. ($\Rightarrow$) Suppose we are given a 4-list coloring c for G'. Since no vertex in G_S contains color z in its list, the restriction of c to G_S is a 3-coloring with colors $\{x, y, a\}$ where the root is not colored using a by the definition of its list. From Lemma 4.2 it follows that there is a leaf v of G_S such that $c(v) = a$. This leaf is connected to all vertices in some S_{i^*} in Step 4, which implies that none of the vertices in S_{i^*} are colored using a. Therefore all vertices in S_{i^*} are colored using x and y. Similarly the gadget G_T has at least one leaf v such that $c(v) = a$, note that this must be a leaf with an odd index by the definition of the lists in Step 5. Therefore there exists T_{j^*} where all vertices are colored using x,y

or z. Thereby in $S_{i^*} \cup T_{j^*}$ only three colors are used, such that S_{i^*} is colored using only two colors. Using Lemma 4.5 and the fact that $G'[S_{i^*} \cup T_{j^*}]$ is isomorphic to the graph obtained from G_{i^*,j^*} by replacing triangles by triangular gadgets, we conclude that X_{i^*,j^*} has a proper 2-3-coloring.

($\Leftarrow$) Suppose $c\colon V(G_{i^*,j^*}) \to \{x,y,z\}$ is a proper 2-3-coloring for X_{i^*,j^*}. We will construct a 4-list coloring $c'\colon V(G') \to \{x,y,z,a\}$ for G'. For u_k with $k \in [m]$ in instance X_{i^*,j^*} let $c'(s_k^{i^*}) := c(u_k)$ and for v_ℓ for $\ell \in [3n]$ let $c'(t_\ell^{j^*}) := c(v_\ell)$. Let $c'(s_\ell^i) := a$ for $i \neq i^*$ and $\ell \in [n]$, furthermore let $c'(t_\ell^j) := z$ for $j \neq j^*$ and $\ell \in [3m]$. For triangular gadgets in T_{j^*} the coloring c' defines all corners to have distinct colors; by Lemma 4.5 we can color the inner vertices consistently using $\{x,y,z\}$. For T_j with $j \in [t']$ and $j \neq j^*$, the corners of triangular gadgets have color z and we can now consistently color the inner vertices using colors x, y, and a.

The leaf of gadget G_S that is connected to S_{i^*} can be colored using a. Every other leaf can use both x and y, so we can properly 3-color the leaves such that one leaf has color a. From Lemma 4.3 it follows that we can consistently 3-color G_S such that the root r_S does not receive color a, as required by $L(r_S)$. Similarly, in triangular gadgets in T_{j^*} the inner vertices do not have color a. As such, leaf $2j^* - 1$ of G_T can be colored using a and we color leaf $2j^*$ with y. For $j \in [t']$ with $j \neq j^*$ color leaf $2j-1$ with z and leaf $2j$ using y. Now the leaves of G_T are properly 3-colored and one is colored a. It follows from Lemma 4.5 that we can color G_T such that the root is not colored a. This completes the 4-list coloring of G'. ◁

It remains to bound the number of vertices of the constructed graph G'. Observe that a treegadget has at least as many leaves as its corresponding binary tree, therefore the graph G' has at most

$$\underbrace{mt'}_{|S|} + \underbrace{12nt'}_{|T|} + \underbrace{6t'}_{|G_S|} + \underbrace{12t'}_{|G_T|} = \mathcal{O}(t' \cdot (m+n)) = \mathcal{O}(\sqrt{t} \max |X_{i,j}|)$$

vertices. Claim 4.8 shows that we have given a degree-2 cross-composition into 4-List Coloring. By Theorem 2.6, and Lemma 4.6 it follows that 4-List Coloring does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$, unless NP $\subseteq$ coNP/poly. To prove the theorem about 4-Coloring, we give a linear parameter transformation from 4-List Coloring. Given an instance (G, L) for 4-List Coloring where every list is a subset of $\{c_1, c_2, c_3, c_4\}$ we construct graph G' for 4-Coloring. Initialize G' as G. We now add new vertices to G' to simulate the lists. Add a clique on 4 vertices $\{c_1, c_2, c_3, c_4\}$. If for any vertex v in G', some color is not contained in $L(v)$, connect v to the vertex corresponding to this color. As proper colorings of the resulting graph correspond to proper list colorings of G', the resulting graph is 4-colorable if and only if there is a YES-instance among the inputs. Theorem 4.7 now follows from Theorem 2.8. ◀

5 Planar List Coloring

Since most of the results obtained for 4-COLORING will also carry over to LIST COLORING, we now consider a slightly different problem, which is list coloring on planar graphs:

PLANAR LIST COLORING	**Parameter:** The number of vertices n
Input: A pair (G, L) where G is an undirected planar graph and color function $L : V(G) \to \mathcal{P}(\mathbb{N})$ that maps each vertex v to a list of colors $L(v)$.	
Question: Is there a proper coloring $c\colon V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V(G)$ and furthermore such that $c(u) \neq c(v)$ for every edge $\{u, v\} \in E(V)$?	

Storing the graph does not take a quadratic amount of space here, since planar graphs have a linear number of edges. However, we need to store a list of colors for each vertex, that may have an arbitrary length.

▶ **Theorem 5.1.** PLANAR LIST COLORING *parameterized by the number of vertices n has a kernel of size $\mathcal{O}(n \log n)$.*

Proof. We present the following simple reduction rule, to reduce the length of the list of a vertex, depending on its degree.

RM Let v be a vertex of the graph with a list of colors with length k and degree d_v. If $k > d_v + 1$, remove vertex v from the graph.

The following lemma shows that this reduction rule is valid.

▷ **Lemma 5.2.** *If instance (G, L) is transformed into (G', L) by applying rule* RM *on vertex v^*, then G has a coloring respecting the lists L if and only if G' has a coloring respecting the lists L.*

Proof. Suppose instance G has a proper coloring respecting the lists L. Since G' is a subgraph of G we can simply use the same coloring for G'.

Suppose G' has a proper coloring c respecting the lists in L. For all vertices except v^*, we use this same coloring in G to obtain a partial coloring respecting the lists. It remains to color vertex v^*. Since $L(v^*)$ has more colors than v^* has neighbors, there is at least one color in $L(v^*)$ that is not used by any of its neighbors, pick such a color to complete the proper list coloring of G. ◁

We repeatedly apply rule RM for all vertices in the graph, until it can no longer be applied. Note that this can be done in polynomial time, since checking if the rule can be applied takes polynomial time and each time it is applied the number of vertices decreases by one. After this process the remaining colors are relabeled from 1 to m, where m is the number of remaining colors. It is clear that relabeling does not influence the answer.

The graph is planar, which implies that the number of edges is linear in the number of vertices. Therefore, $\mathcal{O}(n \log n)$ storage is sufficient for the structure of the graph using

an adjacency list encoding. However, we also need to take the lists of colors into account. In a planar graph, the total number of edges is bounded by $3n$, as proven in Corollary 4.2.10 in [9]. Hereby, the total degree of all vertices is bounded by $6n$. After applying reduction rule RM for every vertex, the length of the list of any remaining vertex v is bounded by $d_v + 1$, such that

$$\text{Total length of all lists } \leq \sum_{v \in V} d_v + 1 \leq 7 \cdot n.$$

Each element of the list also needs a small amount of storage, but since there are at most $7n$ records in total, there are at most $7n$ different colors, implying that only $\mathcal{O}(\log n)$ bits are sufficient to store a single color. Therefore, the total size of all lists is bounded by $\mathcal{O}(n \log n)$ bits. ◀

6 Hamiltonian Cycle

In this section we will show that HAMILTONIAN CYCLE does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$, unless NP $\subseteq$ coNP/poly, by giving a degree-2 cross-composition. It is easy to give a linear parameter reduction from DIRECTED HAMILTONIAN CYCLE to HAMILTONIAN CYCLE. Therefore, we will first prove that DIRECTED HAMILTONIAN CYCLE does not have a generalized kernel of sub-quadratic size, unless NP $\subseteq$ coNP/poly.

(DIRECTED) HAMILTONIAN CYCLE	**Parameter:** The number of vertices n
Input: A (directed) graph G.	
Question: Does there exist a (directed) cycle in G that visits every vertex in $V(G)$ exactly once?	

The construction will make extensive use of the path gadget depicted in Figure 8. Such a gadget has the following property, that will be very useful for giving a cross-composition.

▶ **Lemma 6.1.** *If a directed graph G contains a path gadget as an induced subgraph, such that the remainder of G only connects to the path gadget at vertices* IN^0 *and* IN^1*, then any directed Hamiltonian cycle in G traverses the gadget via Path* 0 *or Path* 1*, as depicted in Figure 8.*

Proof. Any Hamiltonian cycle in G' should visit the center vertex of the path gadget. Since IN^0 and IN^1 are its only two neighbors in G', the only option is to visit them consecutively, Path 0 and Path 1 are the only two options to do this. ◀

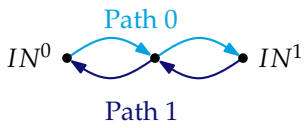


Figure 8. Path gadget.

The following problem will be used as a starting problem for the degree-2 cross-composition.

HAMILTONIAN $s-t$ PATH ON BIPARTITE GRAPHS
Input: An undirected bipartite graph G with partite sets A and B such that $\lvert B\rvert = n = \lvert A\rvert + 1$, together with two distinguished vertices b_1 and b_n that have degree 1.
Question: Does G have a Hamiltonian path from b_1 to b_n?

It is known that Hamiltonian path is NP-complete on bipartite graphs [16] and it is easy to see that it remains NP-complete when fixing a degree 1 start- and endpoint.

▶ **Theorem 6.2.** (DIRECTED) HAMILTONIAN CYCLE *parameterized by the number of vertices n does not have a generalized kernel of size $O(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless* NP $\subseteq$ coNP/poly.

Proof. We first define an equivalence relation $\mathcal{R}$ on instances of HAMILTONIAN $s-t$ PATH ON BIPARTITE GRAPHS. Let two instances be equivalent under $\mathcal{R}$, when they have the same number of vertices. By definition, their independent sets now also have the same size. It is easy to verify that $\mathcal{R}$ is a polynomial equivalence relation.

Suppose we are given t instances $X_1, \ldots, X_t$. Duplicate one of the input instances until $\sqrt{t}$ is an integer. This will multiply the number of instances by at most four and it does not affect the degree-2 cross-composition. Thereby we can relabel the given input instances as $X_{i,j}$, where $i, j \in [\sqrt{t}]$.

Each instance $X_{i,j}$ consists of a graph $G_{i,j}$ and a partition of its vertex set into independent sets $A^*_{i,j}$ and $B^*_{i,j}$. Let $|A^*_{i,j}| = m$ and $|B^*_{i,j}| = n = m + 1$ for each $i, j \in [\sqrt{t}]$. For each instance, label all elements in $A^*_{i,j}$ as $a^*_1, \ldots, a^*_m$ and all elements in $B^*_{i,j}$ as $b^*_1, \ldots, b^*_n$ such that b^*_1 and b^*_n have degree 1.

Using the following steps, we create an instance G' of DIRECTED HAMILTONIAN CYCLE that acts as the logical OR of the inputs.

1. First of all construct $\sqrt{t}$ groups of m path gadgets each. Refer to these groups as A_i, for $i \in [\sqrt{t}]$, and label the gadgets within group A_i as $a^i_1, \ldots, a^i_m$. Let the union of all created groups A_i be named A.

2. Similarly, construct $\sqrt{t}$ groups of n path gadgets each. Refer to these groups as B_j, for $j \in [\sqrt{t}]$, and label the gadgets within group B_j as $b^j_1, \ldots, b^j_n$. Let B be the union of all B_j for $j \in [\sqrt{t}]$.

3. For every input instance $X_{i,j}$, for each edge $\{a^*_k, b^*_\ell\}$ in $G_{i,j}$ with $k \in [m]$, $\ell \in [n]$, do the following.
   - Add an arc from IN^0 of a^i_k to IN^1 of b^j_ℓ
   - and an arc from IN^0 of b^j_ℓ to IN^1 of a^i_k.

If some $X_{i,j}$ has a Hamiltonian $s-t$ path, it can be mimicked by the combination of A_i and B_j, where for each vertex in $X_{i,j}$ we traverse its path gadget in G', following path 1. The following construction steps are needed to extend such a path to a Hamiltonian cycle in G'. Therefore, we create a path through each group A_i for $i \in [\sqrt{t}]$, that traverses all gadgets in this group from a^i_1 up to a^i_m, using Path 0. For groups in B, a similar path is created.

4. Add an arc from the IN^1 terminal of a^i_ℓ to the IN^0 terminal of $a^i_{\ell+1}$ for all $\ell \in [m-1]$ and all $i \in [\sqrt{t}]$.

5. Similarly, add an arc from the IN^1 terminal of b^i_ℓ to the IN^0 terminal of $b^i_{\ell+1}$ for all $\ell \in [n-1]$ and all $i \in [\sqrt{t}]$.

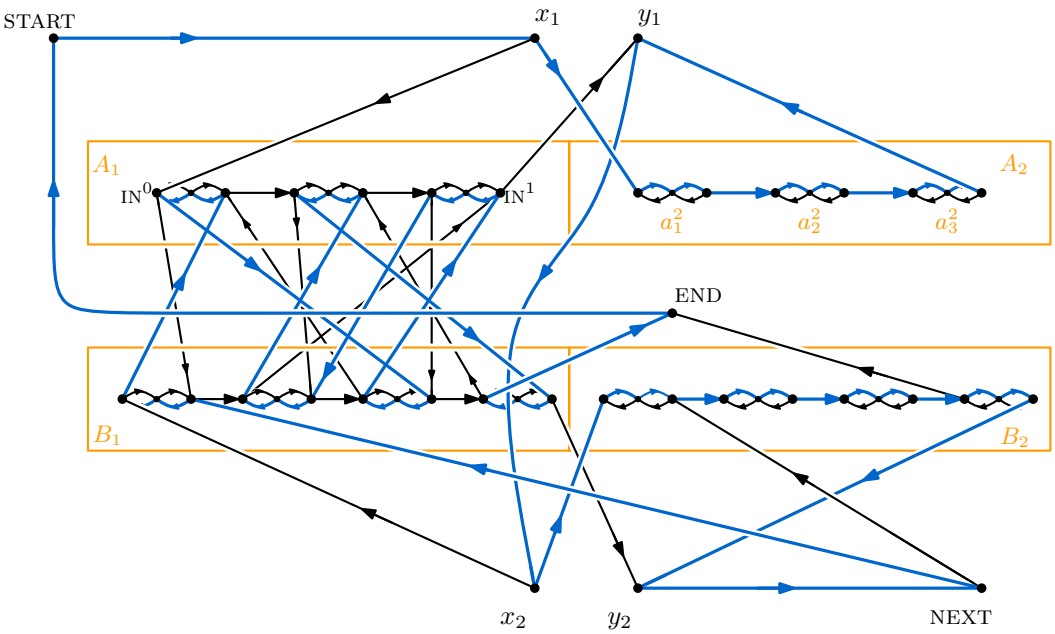


Figure 9. The general structure of the created graph G' , when given 4 inputs (such that $r = 1$) with $n = 3$ and $m = 4$. A possible Hamiltonian Cycle is shown in blue.

We create an additional structure that allows us to select exactly $\sqrt{t} - 1$ groups from A and $\sqrt{t} - 1$ groups from B, for which all gadgets are traversed in order via Path 0. This leaves exactly one group A_i and one group B_j to be traversed using Path 1.

6. Add a vertex START and a vertex END and the arc $(\text{END}, \text{START})$.

7. Let $r := \sqrt{t} - 1$, add $2r$ tuples of vertices, x_i, y_i for $i \in [2r]$ and connect START to x_1. Furthermore, add the arcs (y_i, x_{i+1}) for $i \in [2r - 1]$.

8. For $i \leq r$ we add arcs from x_i to the IN^0 terminal of the gadgets $a_1^j, j \in [\sqrt{t}]$. Furthermore we add an arc from IN^1 of a_m^j to y_i for all $j \in [\sqrt{t}]$ and $i \in [r]$. When $i > r$ add arcs from x_i to the IN^0 terminal of b_1^j for $j \in [\sqrt{t}]$ and connect IN^1 of b_n^j to y_i.

9. Add a vertex NEXT and the arc (y_{2r}, NEXT) and an arc from NEXT to the IN^1 terminal of all gadgets b_1^j for $j \in [\sqrt{t}]$.

10. Furthermore, add arcs from IN^0 of all gadgets b_n^j to END for $j \in [\sqrt{t}]$. So for each B_j, exactly one vertex has an outgoing arc to END and one has an incoming arc from NEXT.

This completes the construction of G'. A sketch of G' is shown in Figure 9.

In Steps 4 and 5 of the construction a path through each group of gadgets in A and B is created. The following lemma shows how this path is used by a Hamiltonian cycle.

▷ **Lemma 6.3.** *When any Hamiltonian cycle in G' enters the path gadget of a_1^i at IN_0, the cycle then visits the gadgets of $a_2^i, a_3^i, \ldots, a_m^i$ in order without visiting other vertices in between. Similarly, if any Hamiltonian cycle in G' enters the path gadget of b_1^j at IN_0, the cycle then visits the gadgets of $b_2^j, b_3^j, \ldots, b_n^j$ in order without visiting other vertices in between.*

Proof. Consider a Hamiltonian cycle in G' that enters path gadget a_1^i at IN_0. By Lemma 6.1 the cycle follows Path 0 and continues to the IN^1 terminal of the path gadget. Since that terminal has only one outgoing arc leaving the gadget, which goes to the IN_0 terminal of a_2^i, it follows that the cycle continues to that path gadget and enters it at IN_0. By repeating this argument, the cycle must continue to traverse gadgets a_3^i up to a_m^i using Path 0 and without visiting other vertices in between. The proof when entering group B_j at the vertex IN_0 of b_1^j is equivalent. ◁

The following lemma shows that if G' contains a directed Hamiltonian cycle, the subpath from x_1 to y_{2r} will be used to traverse exactly $r-1$ groups in A and $r-1$ groups in B.

▷ **Lemma 6.4.** *Let C be a directed Hamiltonian cycle in G', such that its first arc is $\{\text{START}, x_1\}$. There are indices $i^*, j^* \in [\sqrt{t}]$ such that subpath $C_{x_1,y_{2r}}$ of the cycle between x_1 and y_{2r} contains exactly the vertices*

$$\overline{A_{i^*}} \cup \overline{B_{j^*}} \cup \{x_i, y_i \mid i \in [2r]\}$$

where $\overline{A_{i^*}}$ contains all vertices of all gadgets in A_i for $i \neq i^*$ and similarly $\overline{B_{j^*}}$ contains all vertices of all gadgets in B_j for $j \neq j^*$.

Proof. We will first show that when the cycle reaches any x_i for $i \in [r]$, it traverses exactly one group A_ℓ with $\ell \in [r+1]$ and continues to y_j and x_{j+1} for some $j \in [r]$, without visiting other vertices in between. Similarly, when the cycle reaches any x_i for $r < i \leq 2r$, it traverses exactly one group B_ℓ with $\ell \in [r+1]$ and continues to y_j for some $r < j \leq 2r$. For $j < 2r$, the cycle then continues to x_{j+1}, for $j = 2r$ the cycle reached y_{2r}, which is the last vertex of this subpath.

By Step 8 in the construction, all outgoing arcs of any x_i for $i \in [r]$ connect to gadgets a_1^ℓ for some $\ell \in [\sqrt{t}]$. So for any x_i in the cycle there must be a unique $\ell \in [\sqrt{t}]$ such that the edge from x_i to the IN^0 terminal of a_1^ℓ is in C. By Lemma 6.3 the cycle visits all vertices in A_ℓ, and no other vertices, before reaching gadget a_m^ℓ, which is traversed by Path 0 to get to OUT^0 of this gadget. The only neighbors of OUT^0 of gadget a_m^ℓ lying outside this gadget are of type y_j for $j \in [r]$. As such, the cycle must visit some y_j next, and its only outgoing arc goes to x_{j+1}.

The proof for $i > r$ is similar. As such, visiting x_i for $i \in [r]$ results in visiting all vertices of exactly one group in A before continuing via y_j to some x_{j+1} without visiting any vertices in between. Visiting x_i for $r < i \leq 2r$ results in visiting all vertices of exactly

one group in B and returning via y_j to either the end of the subpath ($j = 2r$) or some x_{j+1}.

Every vertex x_i for $i \in [2r]$ must be visited by C, it remains to show that it is visited in subpath $C_{x_1,y_{2r}}$. Suppose there exists an x_i for $i \in [2r]$ such that x_i is not visited in the subpath from x_1 to y_{2r}. As we have seen above, visiting some x_i results in visiting all vertices in some group in A or B, continued by visiting some y_j for $j \in [2r]$. Note that no other vertices are visited in between. Hereby, y_j is not in subpath $C_{x_1,y_{2r}}$. This implies $j \neq 2r$ and thus the next vertex in the cycle is x_{j+1}. So, for x_i not in subpath $C_{x_1,y_{2r}}$, one can find a new vertex x_{j+1} (where $j+1 \neq i$), such that x_{j+1} is also not in subpath $C_{x_1,y_{2r}}$. Note that we can not create a loop, by visiting a vertex x_i seen earlier, as this would not yield a Hamiltonian cycle in G'. For example, the vertex START would never be visited. This is however a contradiction since we only have finitely many vertices x_i.

Thus in subpath $C_{x_1,y_{2r}}$, exactly r groups of A are visited and exactly r groups of B are visited, and no other vertices than specified. This leaves exactly one group A_{i^*} and one group B_{j^*} unvisited in $C_{x_1,y_{2r}}$. ◁

In the lemma above, one group in A and one group in B remain unvisited. The next lemma shows that these groups are visited in the subpath between NEXT and END. This result will be useful to show that one of the input instances must have a Hamiltonian path if G' has a Hamiltonian cycle.

▷ **Lemma 6.5.** *Let C be a Hamiltonian cycle in G', such that its first arc is $\{\text{START}, x_1\}$. Let i^* and j^* satisfy the conditions of Lemma 6.4. Then cycle C visits $b_1^{j^*}$ before $b_n^{j^*}$. Moreover, the subpath of the cycle $C_{b_1^{j^*},b_n^{j^*}}$ between terminal IN^1 of $b_1^{j^*}$ and OUT^1 of $b_n^{j^*}$ (inclusive) contains all vertices of the gadgets in A_{i^*} and B_{j^*} and no others.*

Proof. Vertex NEXT is visited directly after y_{2r}. The arc from NEXT to gadget b_1^ℓ must be in the cycle for some $\ell \in [\sqrt{t}]$, since these are the only outgoing edges of NEXT, constructed in Step 9. By Lemma 6.4, all gadgets in all B_j for $j \neq j^*$ are visited in the path from x_1 to y_{2r}, and thus should not be visited after vertex NEXT. Therefore, the edge from NEXT to gadget $b_1^{j^*}$ is in the cycle, which also implies that $b_1^{j^*}$ is visited before $b_n^{j^*}$.

It is easy to see that $\{\text{END}, \text{START}\}$ is the last arc in C. By considering the incoming arcs of END it follows that some arc from terminal OUT^1 of b_n^ℓ to END for $\ell \in [\sqrt{t}]$ is in the cycle. Since the vertices in gadgets b_n^ℓ for $\ell \neq j^*$ are already visited in $C_{x_1,y_{2r}}$ by Lemma 6.4, it follows that $\{b_n^{j^*}, \text{END}\}$ is in C.

By Lemma 6.4, none of the terminals of gadgets in A_{i^*} and B_{j^*} is visited in the subpath $C_{x_1,y_{2r}}$ or equivalently in the subpath $C_{\text{START,NEXT}}$. Since C is a Hamiltonian cycle these vertices must therefore be visited in $C_{\text{NEXT,START}}$, which is equivalent to saying that $C_{b_1^{j^*},b_n^{j^*}}$ must contain all vertices in $A_{i^*} \cup B_{j^*}$. It is easy to see that this subpath cannot contain any other vertices, as all other vertices are present in $C_{\text{START,NEXT}}$ or $C_{\text{END,START}}$. ◁

Using the previous lemmas we can now show that G' acts as a logical OR of the given inputs.

▷ **Claim 6.6.** *Graph G' has a directed Hamiltonian cycle if and only if at least one of the instances $X_{i,j}$ has a Hamiltonian $s-t$-path.*

Proof. ($\Leftarrow$) Suppose G' has a Hamiltonian cycle C. By Lemma 6.5 there exist $i^*, j^* \in [\sqrt{t}]$ such that the subpath of C from gadget $b_1^{j^*}$ to $b_n^{j^*}$ visits exactly the gadgets in $A_{i^*} \cup B_{j^*}$. Since gadget $b_1^{j^*}$ is entered at terminal IN^1, it is easy to see that all gadgets are traversed using Path 1. We now construct a Hamiltonian path P for instance $X_{i^*j^*}$. Let $\{a_k^*(i^*j^*), b_\ell^*(i^*j^*)\} \in P$ if the arc from OUT^1 of $a_k^{i^*}$ to IN^1 of $b_\ell^{j^*}$ is in C. Similarly let $\{b_k^*(i^*j^*), a_\ell^*(i^*j^*)\} \in P$ if the arc from OUT^1 of $b_\ell^{j^*}$ to IN^1 of $a_k^{i^*}$ is in C, where $k \in [m]$ and $\ell \in [n]$. Using that every gadget is visited exactly once via Path 1 in C, we see that C is a Hamiltonian path.
($\Rightarrow$) Suppose $X_{i^*j^*}$ has a Hamiltonian $s-t$ path P. Then we create a Hamiltonian cycle C, for each vertex a_ℓ^* from instance $X_{i^*j^*}$ in P we add Path 1 in path gadget $a_\ell^{i^*}$ to C and for each vertex b_ℓ^* we add Path 1 in path gadget $b_\ell^{j^*}$ to C. Let P be ordered such that b_1^* is its first vertex. Now if a_k^* is followed by b_ℓ^* in P, the arc from terminal OUT^1 of $a_k^{i^*}$ to IN^1 of $b_\ell^{j^*}$ is added to C. Similarly, if a vertex b_ℓ^* is followed by a_k^* in P, the arc from terminal OUT^1 of $b_\ell^{j^*}$ to IN^1 of $a_k^{i^*}$ will be added to C. Now the subpath $C_{b_1^{j^*}, b_n^{j^*}}$ contains all terminals in all gadgets in $A_{i^*} \cup B_{j^*}$.

From $b_n^{j^*}$ the cycle goes to END, then to START and to x_1. To visit all groups A_i for $i \neq i^*$ and B_j for $j \neq j^*$, do the following.

- From x_i where $i \leq i^*$, the cycle continues to gadgets a_1^i, then to $a_2^i, a_3^i, \ldots, a_m^i$ following Path 0, and continue to y_i, x_{i+1}.
- From x_i where $i^* \leq i \leq r$ it goes to $a_1^{i+1}, a_2^{i+1}, \ldots, a_n^{i+1}$ and continues with y_i, x_{i+1}.
- Similarly, from x_i where $r \leq i < j^*$, go through gadgets $b_1^i, \ldots, b_n^i$ and continue to y_i, x_{i+1}.
- From x_i where $j^* \leq i \leq 2r$, go to gadgets $b_1^{i+1}, \ldots, b_n^{i+1}$ and continue to y_i, for $i \neq 2r$ then add the arc (y_i, x_{i+1}).

From y_{2r}, continue to NEXT, after which the edge $(\text{NEXT}, b_1^{j^*})$ closes the cycle. By definition, no vertex is visited twice, so it remains to check that every vertex of G' is in the cycle. For vertices START, NEXT, END and all vertices x_i, y_i, z_i this is obvious. All vertices in A_i and B_j where $i \neq i^*$ and $j \neq j^*$ are in the cycle between some x_ℓ and y_ℓ. All vertices in A_{i^*} and B_{j^*} are visited since P was a Hamiltonian path on these vertices. ◁

The number of vertices of G' is

$$\underbrace{3(m+n)\sqrt{t}}_{|A \cup B|} + \underbrace{4(\sqrt{t}-1)}_{x_i,\, y_i} + 3 = \mathcal{O}(\sqrt{t} \cdot (m+n))$$

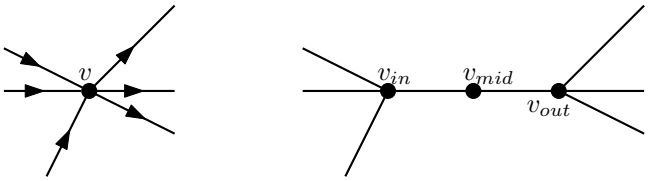


Figure 10. Reduction from DIRECTED HAMILTONIAN CYCLE to HAMILTONIAN CYCLE.

therefore this is a degree-2 cross composition. It is possible to give a linear parameter transformation from DIRECTED HAMILTONIAN CYCLE to HAMILTONIAN CYCLE, using the NP-completeness proof by Karp in [19]. Suppose we are given $G = (V, A)$ for directed Hamiltonian cycle, we construct $G' = (V', E')$ by $V' := \{v_{in}, v_{mid}, v_{out} \mid v \in V\}$ and $E' := \{\{v_{in}, v_{mid}\}, \{v_{mid}, v_{out}\}\} \cup \{\{u_{out}, v_{in}\} \mid (u, v) \in A\}$ as shown in Figure 10. It is easy to verify that this is a linear parameter transformation. Using Theorems 2.6 and 2.8 we conclude that (DIRECTED) HAMILTONIAN CYCLE does not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless NP $\subseteq$ coNP/poly. ◀

7 Dominating Set

In this section, we will show that DOMINATING SET and CONNECTED DOMINATING SET do not have a kernel of sub-quadratic size, when parameterized by the number of vertices.

DOMINATING SET	**Parameter:** The number of vertices n
Input: An undirected graph $G = (V, E)$ and integer $k \in \mathbb{N}$	
Question: Does there exist a set $S \subseteq V(G)$ with $\lvert S\rvert \leq k$, such that every vertex $v \in V(G) \setminus S$ has at least one neighbor in S?	

We will also consider the CONNECTED DOMINATING SET problem which is defined similarly except that there is an extra constraint that $G[S]$ is connected.

A set S with the described properties is called a *(connected) dominating set* in G. We say that a vertex u is dominated by v (with respect to dominating set S) if u is a neighbor of v and v is contained in the dominating set.

We will prove the sparsification lower bound using a degree-2 cross-composition, starting from a variation of the COLORED RED-BLUE DOMINATING SET problem (COL-RBDS) as described by Dom *et al.* in [10].

EQUAL-SIZED COLORED RED/BLUE DOMINATING SET (EQ-COL-RBDS)

Input: A bipartite graph $G = (R \cup B, E)$, where R is partitioned into k subsets $R_1, \ldots, R_k$, such that $\lvert R_1\rvert = \lvert R_2\rvert = \ldots = \lvert R_k\rvert$.

Question: Is there a set $S \subseteq R$ such that for each $i \in [k]$ the set S contains exactly one vertex of R_i and every vertex in B is adjacent to at least one vertex from S?

We will think of the vertices in set R_i as having color i such that the question is if there exists a set $S \subseteq R$ that has exactly one vertex of each color and every vertex in B is adjacent to at least one vertex in S.

▶ **Lemma 7.1.** EQ-COL-RBDS *is NP-complete.*

Proof. It is easy to give a reduction from COLORED RBDS, without requiring that all color sets have the same size. This problem was proven to be NP-complete by Dom *et al.* in [10]. Let an input instance on n vertices for Col-RBDS be given, let the largest color set have size ℓ. We then add isolated vertices to all color other sets until their size is exactly ℓ. Since the added vertices are isolated, they do not influence the size of a minimum RBDS in the graph. Note that both the number of color sets and their size are bounded by n, as such we have created a graph with at most n^2 vertices, which is polynomial. ◀

Using EQ-COL-RBDS as our starting problem, we can give a degree-2 cross-composition.

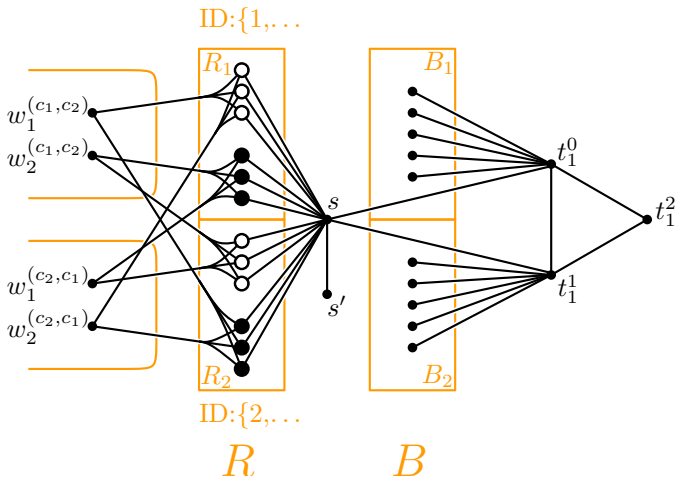


Figure 11. A sketch of G, where $t' = 2$, $m = 6$ and $n = 5$. Thereby K should be 5 and $W_{(c_1,c_2)}$ should contain 10 vertices. In this example we show the constructed graph when choosing $K = 1$ for simplicity. Edges from R to B are left out for simplicity.

▶ **Theorem 7.2.** DOMINATING SET *and* CONNECTED DOMINATING SET *parameterized by the number of vertices n do not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless* NP $\subseteq$ coNP/poly.

Proof. We can use the same degree-2 cross-composition to prove both results.

Define a polynomial equivalence relation $\mathcal{R}$ by first of all letting all instances where there is a vertex in B of degree 0 be in the same class, note that these are always no-instances. Let 2 instances $(G = (R \cup B), k)$ and $(G' = (R' \cup B'), k')$ of EQ-COL-RBDS be equivalent if $|R| = |R'|$, $|B| = |B'|$ and $k = k'$. It is easy to see that $\mathcal{R}$ indeed is a polynomial equivalence relation.

Suppose we are given t instances of EQ-COL-RBDS, such that $\sqrt{t}$ and $\log \sqrt{t} \in \mathbb{N}$ and such that all given instances are in the same equivalence class of $\mathcal{R}$. Let $t' := \sqrt{t}$. If these instances are from the class where B contains a vertex of degree 0, output a constant size no-instance.

Otherwise, label the given instances as $X_{i,j}$ with $i, j \in [t']$. Let instance $X_{i,j}$ have graph $G_{i,j}$, which is bipartite with vertex set $R^*_{i,j} \cup B^*_{i,j}$. Let $|R^*_{i,j}| = m$ and $|B^*_{i,j}| = n$ and let $R^*_{i,j}$ be partitioned into k color classes $R^{*p}_{i,j}$ for all $i, j \in [t']$ and $p \in [k]$. Label all vertices in $R^{*p}_{i,j}$ as $r^*_{p,q}(i,j)$ with $p \in [k]$ and $q \in [m/k]$, which means that this vertex is the q'th vertex of color p from instance $X_{i,j}$. Label vertices in $B^*_{i,j}$ as $b^*_1(i,j), \ldots, b^*_n(i,j)$ arbitrarily. We now create an instance (G, k) for DOMINATING SET using the following steps. A sketch of G can be found in Figure 11.

1. Add vertices $r^i_{p,q}$ for $p \in [k], q \in [m/k]$ and $i \in [t']$. The dominating set problem does not use colored instances, however we will remember the color of these

vertices for simplicity. Let vertex $r^i_{p,q}$ have color p, for $i \in [t']$, $q \in [m/k]$ and $p \in [k]$. Define $R_i := \{r^i_{p,q} \mid p \in [k], q \in [m/k]\}$ and let $R := \bigcup_{i\in[t']} R_i$. Give every set R_i a unique identifier $ID(R_i)$, which is a subset of $K := 2 + k + \log t'$ numbers in the range $[2K]$. We can construct this identifier by considering the binary representation b of $i-1$, which has length $\log t$. Consider the string $b\bar{b}$, where $\bar{b}$ contains a zero at a certain position if and only if b contains a one in this position. String $b\bar{b}$ has exactly $\log t$ positions with a 1. Now if the j'th position contains a 1, let $j \in ID(R_i)$. This already ensures that every R_i gets a unique ID containing $\log t$ integers in the range $[2 \log t']$. Then add $k+2$ distinct integers from the range $2\log t + 1, \ldots, 2\log t + 2k + 4$ to the ID.

2. Add vertices b^j_ℓ for $\ell \in [n]$ and $j \in [t']$. Define B_j and B as $B_j := \{b^j_\ell \mid \ell \in [n]\}$ and $B := \bigcup_{j\in[t']} B_j$.

3. Add edges between the vertices $r^i_{p,q}$ and b^j_ℓ for $p \in [k], q \in [m/k]$ and $i, j \in [t']$ if $r^*_{p,q}(i,j)$ is connected to $b^*_\ell(i,j)$ in instance $X_{i,j}$. This ensures that the graph induced by $R_i \cup B_j$ is exactly $G_{i,j}$ and the coloring of vertices in R_i matches the coloring of $R^*_{i,j}$.

4. Add vertices s' and s and edge $\{s', s\}$. Furthermore, add edges between s and all vertices in R. The degree-1 vertex s' ensures there is a minimum dominating set containing s, which covers all vertices in R "for free".

5. In a similar way as given by Dom *et al.* in [10], for every pair of colors $(c_1, c_2) \in \{1, \ldots, k\} \times \{1, \ldots, k\}$ with $c_1 \neq c_2$ we add a vertex set

$$W_{(c_1,c_2)} = \{w_1^{(c_1,c_2)}, \ldots, w_{2K}^{(c_1,c_2)}\}.$$

   For $x \in [2K]$ connect $w_x^{(c_1,c_2)}$ to all vertices of color c_1 in R_i if $x \in ID(R_i)$, otherwise connect $w_x^{(c_1,c_2)}$ to all vertices of color c_2 in R_i. This construction is used to choose which R_i is part of a solvable input instance $X_{i,j}$ for some $j \in [t']$. This idea is formalized in Lemmas 7.5 and 7.6.

6. Then, add $\log t'$ triangles, with vertices $\{t^0_\ell, t^1_\ell, t^2_\ell\}$ for $\ell \in [\log t']$. Connect t^0_ℓ to all vertices in B_j if the ℓ'th bit of j equals 0, connect t^1_ℓ to all vertices in B_j if the ℓ'th bit of j equals 1. Define T to be the union of all these triangles. By choosing exactly one of the vertices t^0_ℓ or t^1_ℓ in a dominating set for each ℓ, all groups B_j except one are dominated automatically. The non-dominated one should then be part of a solvable input instance.

7. Finally, add the edges $\{\{s, t^i_\ell\} \mid \ell \in [\log t'], i \in \{0,1\}\}$. Note that t^2_ℓ is not a neighbor of s and that t^0_ℓ and t^1_ℓ are its only neighbors. This final step of the construction ensures that every vertex in T contained in the dominating set will

have s as a neighbor in the dominating set, such that there is always a minimum dominating set that is connected. This result will be formalized in Claim 7.7.

We now make the following observations.

▷ **Lemma 7.3.** *If G has a dominating set D, then it also has a dominating set D' of size at most $|D|$ that does not contain any vertices from B.*

Proof. Suppose we are given a minimum dominating set D of G, where vertex $v \in B$ is present. In any dominating set, s or s' must be present. If s' is present and s is not, we replace s' by vertex s, and still obtain a valid dominating set of the same size. As such, all vertices in R are now dominated by s. Vertices t^0_ℓ and t^1_ℓ with $\ell \in [\log t']$ are dominated by s. Since t^2_ℓ only has neighbors t^1_ℓ and t^0_ℓ, at least one of these three vertices is present in D for every $\ell \in [\log t']$, hereby every vertex in T has a neighbor in D.

Since B is an independent set in G, the vertex v does not dominate other vertices in B. Since the polynomial equivalence relation ensures that there are no isolated vertices in B, vertex v has at least one neighbor u in R. We can safely replace v by u to obtain a valid dominating set that has the same size as D and does not contain any vertices from B. ◁

▷ **Lemma 7.4.** *Any dominating set of G of size at most $k + 1 + \log t'$ contains at least $1 + \log t'$ vertices from $\{s, s'\} \cup \{t^0_\ell, t^1_\ell, t^2_\ell \mid \ell \in [\log t']\}$ and thus contains at most k vertices from R.*

Proof. In a dominating set D of G, at least $\log t'$ vertices are needed from T, since t^2_ℓ only has neighbors t^1_ℓ and t^0_ℓ, so one of these three vertices must be in D for each $\ell \in [\log t']$. Furthermore at least one of the vertices s' or s must be present, therefore there are $1 + \log t'$ vertices in the set that are not from R. ◁

We obtained that a dominating set of size at most $k + 1 + \log t'$ contains at most k vertices from R. This result can now be used to prove that such a dominating set contains exactly one vertex of each color.

▷ **Lemma 7.5.** *Any dominating set of G of size at most $k + 1 + \log t'$ uses exactly one vertex of each color from R.*

Proof. Suppose a dominating set of G of size at most $k + 1 + \log t'$ uses less than k colors from R. If at most $k - 2$ colors are used, there must be two colors c_1 and c_2 that are not present in the set. However, this implies that all $2K$ vertices in $W^{(c_1,c_2)}$ are not dominated by vertices in R and must therefore be in the set. This contradicts the maximum size of the dominating set, since $2K = 2k + 4 + 2\log t'$. So, we are left with the possibility of using $k - 1$ colors. Consider some color c_1 that was not used. Look at another color c_2 that is used exactly once, such a color exists by Lemma 7.4. Suppose the vertex of color c_2 in the dominating set was from set R_i for some $i \in [t']$. Then for any $x \in ID(R_i)$ we have that $w^{(c_1,c_2)}_x$ is not connected to any vertex in the dominating set and therefore must be in

the dominating set itself. Since $ID(R_i)$ contains K numbers, there are $K > k + 1 + \log t'$ vertices in W that are not dominated by R, which contradicts the maximum size of the dominating set. ◁

The lemma above also implies that there are exactly k vertices from R in any dominating set of G that has size at most $k + 1 + \log t'$. It then follows that none of the vertices in W are in the dominating set. We can now show that the vertices chosen from R are all contained in the same set R_i for some i.

▷ **Lemma 7.6.** *For any dominating set D of G of size at most $k + 1 + \log t'$, there exists $i \in [t']$ such that all vertices in $D \cap R$ are contained in set R_i.*

Proof. Suppose there exists two vertices $u, v \in D$ such that $u \in R_i$ and $v \in R_j$ for some $i \neq j$. By Lemma 7.5, u and v have different colors. Suppose u has color c_u and v has color c_v. Since $R_i \neq R_j$, there exists $x \in [2K]$ such that $x \in ID(R_i)$ and $x \notin ID(R_j)$. By Step 5 of the construction, this means that none of the neighbors of vertex $w_x^{(c_u,c_v)}$ are contained in the dominating set. However, this vertex is not in D and therefore D is not a dominating set of G, which is a contradiction. ◁

Using the previous Lemmas, we can prove that G' acts as a logical OR of the given input instances, for both DOMINATING SET and CONNECTED DOMINATING SET.

▷ **Claim 7.7.**

1. *If there is an input X_{i^*,j^*} that has a col-RBDS of size k, then G' has a connected dominating set of size $k + 1 + \log t'$.*
2. *If G' has a (not necessarily connected) dominating set of size $k + 1 + \log t'$, then some input X_{i^*,j^*} has a col-RBDS of size k.*

Proof. **1.** Let X_{i^*,j^*} have a colored RBDS D of size at most k, then we can construct a dominating set D' of G in the following way. For any vertex $r^*_{p,q}$ in D, add vertex $r^i_{p,q}$ to D'.

Furthermore add the vertex s to D'. Then add vertex t^0_ℓ to D' if the q'th bit of j^* is 1, add vertex t^1_ℓ otherwise. Now s' is dominated and all vertices in R have neighbor s in D'. All vertices in B_{j^*} are covered by the vertices in the dominating set from R_{i^*}, since D was a col-RBDS of X_{i^*,j^*}. All vertices in B_j for $j \neq j^*$ have neighbor t^0_ℓ or t^1_ℓ in D' for some $\ell \in [\log t']$, since the bit representation of j must differ from the one of j^* at some position. It now follows from Step 6 of the construction that all vertices in B_j are connected to a vertex in the dominating set.

It remains to verify that all vertices in W have a neighbor in D'. Consider $w_x^{(c_1,c_2)}$ for $x \in [2K]$ and $c_1, c_2 \in [k]$. If $x \in ID(R_{i^*})$, then this vertex is connected to all vertices of color c_1 and exactly one of them is contained in D'. If $x \notin ID(R_{i^*})$, the

vertex $w_x^{(c_1,c_2)}$ is connected to all vertices of color c_2 in R_{i^*} and again one vertex of this color in R_{i^*} is contained in D'. So D' is a dominating set of G and it is easy to verify that $|D'| = k + 1 + \log t'$. Furthermore, D' is constructed in such a way that it is connected. We can show this by proving that every vertex in D' is a neighbor of s, since we chose s in D. Vertices in $D' \cap R$ and $D' \cap T$ are neighbors of s, by Steps 4 and 7 of the construction of G. The vertex s' and vertices from W and B are not contained in D'. Thus, D' is a connected dominating set.

2. Let D' be a dominating set of G of size at most $k + 1 + \log t'$. Using Lemma 7.3 we modify D' such that it chooses no vertices from B, without increasing its size. By Lemma 7.5 and 7.6, D' contains exactly k vertices from R, all from the same R_{i^*} for some i^* and all of different color. D' has size at most $k + 1 + \log t$ of which k are contained in R and one in $\{s, s'\}$. Combined with the fact that for any $\ell \in [\log t']$ vertex t_ℓ^2 has t_ℓ^1 and t_ℓ^0 as its only two neighbors, it follows that exactly one of these three vertices is contained in D' for all ℓ. Therefore D' contains at most one of the vertices t_ℓ^0 or t_ℓ^1 for every $\ell \in [\log t']$.

We can now define $x_\ell \in \{0, 1\}$ for $\ell \in [\log t']$, such that $t_\ell^{x_\ell} \notin D'$ for all $\ell \in [\log t']$. Consider the index $j^* \in [t]$ given by the binary representation $[x_1\, x_2\, \ldots\, x_{\log t'}]_2$. It follows from the bit representation of j^* that the vertices in B_{j^*} are not connected to any of the vertices in $D' \cap T$. Since vertices in B_{j^*} are only adjacent to vertices in R and vertices of T, it follows that every vertex in B_{j^*} has a neighbor in R that is in D'. This implies that every vertex in B_{j^*} has a neighbor in $D' \cap R_{i^*}$. Since $G[R_{i^*} \cup B_{j^*}]$ is isomorphic to the graph of instance X_{i^*,j^*}, it follows that X_{i^*,j^*} has a col-RBDS of size at most k, which are exactly the vertices in $D' \cap R_{i^*}$. ◁

Given t instances, the graph G constructed above has

$$\underbrace{m \cdot t'}_{|R|} + \underbrace{n \cdot t'}_{|B|} + \underbrace{2}_{s,s'} + \underbrace{3 \cdot \log t'}_{|T|} + \underbrace{2\binom{k}{2} \cdot 2K}_{|W|} = \mathcal{O}(\sqrt{t} \max |X_{i,j}|^3)$$

vertices. It is straightforward to construct G in polynomial time. It follows from Claim 7.7 that G has a dominating set of size $k + 1 + \log t'$, if and only if one of the input instances has a col-RBDS of size k. Furthermore, G has a connected dominating set of size $k + 1 + \log t'$ if and only if one of the input instances has a col-RBDS of size k. Therefore we have given a degree-2-cross-composition to (CONNECTED) DOMINATING SET. Using Theorem 2.6 it follows that DOMINATING SET and CONNECTED DOMINATING SET do not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. ◀

This lower bound has immediate implications for the sparsification lower bounds of the dual problems NONBLOCKER and MAX LEAF SPANNING TREE. It is hard to give an intuition about NONBLOCKER, other than that a graph has a NONBLOCKER of size at least k if and only if it has a DOMINATING SET of size at most $n - k$. MAXIMUM LEAF

SPANNING TREE seems to be a more natural problem, where we ask whether a given graph contains a spanning tree with at least k leaves. Using Theorem 7.2 we obtain the following result.

▶ **Corollary 7.8.** MAX LEAF SPANNING TREE *and* NONBLOCKER *parameterized by the number of vertices n do not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$ for any $\varepsilon > 0$, unless* $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

Proof. By definition, a graph has a Nonblocker of size k if and only if is has a Dominating Set of size $n - k$. Thereby it is trivial to give a linear parameter transformation from DOMINATING SET to NONBLOCKER and use Theorems 7.2 and 2.8 to obtain the lower bound.

If a graph has a spanning tree with at most k leaves, the non-leaves of this tree correspond to connected dominating set of G, that has size $n - k$. Vice versa, given a connected dominating set of size $n - k$, we extend it to a spanning tree with k leaves in a straightforward way. Thus, a graph has a connected dominating set of size $n - k$ if and only if it has a spanning tree with k leaves. Thereby it is trivial to give a linear parameter transformation from CONNECTED DOMINATING SET to MAX LEAF SPANNING TREE and use Theorems 7.2 and 2.8 to obtain the lower bound. ◀

Both DOMINATING SET and CONNECTED DOMINATING SET are not believed to be fixed parameter tractable when parameterized by the solution size ([12], [6]), so no kernels are known. NONBLOCKER and MAXIMUM LEAF SPANNING TREE are however fixed parameter tractable and both have a linear size kernel. NONBLOCKER has a kernel of $\frac{5}{3}k + 3$ vertices [7] and MAXIMUM LEAF SPANNING TREE has a kernel of $3.75k$ vertices [14]. Since for these problems input parameter k is bounded by n, Theorem 7.2 shows that we cannot reduce the number of edges of these kernels to $\mathcal{O}(k^{2-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.

8 d-Hypergraph 2-Colorability and d-NAE-Sat

In this section we will give a kernelization lower bound of d-HYPERGRAPH 2-COLORABILITY, by giving a linear parameter reduction starting from d-NAE-SAT. Furthermore we will show that both this problem and d-NAE-SAT have a (generalized) kernel consisting of $\mathcal{O}(n^{d-1})$ edges or clauses.

d-HYPERGRAPH 2-COLORABILITY	**Parameter:** The number of vertices n
Input: A hypergraph $G = (V, E)$, where each hyperedge contains at most d vertices.	
Question: Is there a coloring using $\{Red, Blue\}$ of all vertices in V such that every edge in E contains at least one Red and one Blue vertex.	

d-NOT-ALL-EQUAL SAT	**Parameter:** The number of variables n
Input: A CNF formula F on n variables with at most d literals per clause.	
Question: Is there an assignment for all variables, such that each clause contains at least one true and at least one false literal? If such an assignment exists, we say F if *NAE-satisfiable*.	

8.1 Lower bound

We can show a lower bound on the kernel size of d-NAE-SAT, which we will later use to prove a lower bound for the kernel of d-HYPERGRAPH 2-COLORABILITY. The following result was also shown by Jansen *et al.* in [18].

► **Lemma 8.1.** *Let $d \geq 4$ be an integer. Then d-NAE-SAT parameterized by the number of variables n does not have a generalized kernel of size $O(n^{d-1-\varepsilon})$ for any $\varepsilon > 0$, unless* NP $\subseteq$ coNP/poly.

Proof. It is easy to give a linear parameter reduction from d-CNF-SAT to $(d+1)$-NAE-SAT. By Theorem 2.8 it follows that if $d+1$-NAE-SAT has a generalized kernel of size $\mathcal{O}(n^{d-\varepsilon})$, then also d-CNF-SAT has a generalized kernel of size $O(n^{d-\varepsilon})$. In Theorem 1 in [8] Dell *et al.* prove an even stronger statement than the result we need here. Their theorem implies is that d-CNF-SAT does not have a generalized kernel of size $\mathcal{O}(n^{d-\varepsilon})$

So suppose we are given a Boolean formula F of d-CNF-SAT, with variables $X = x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. We will now create an instance F' of $(d+1)$-NAE-SAT with variables $X' := \{x_1, \ldots, x_n\} \cup y$ and clauses $C'_1, \ldots, C'_m$, where $C'_i := C_i \cup y$ for $1 \leq i \leq m$. It follows that every clause C'_i has size at most $d+1$. It remains to verify that F' is NAE-satisfiable if and only if F is satisfiable.

Suppose F is satisfiable, we can extend a satisfying truth S assignment with $y :=$ *false* to NAE-satisfy F'. Every clause now contains one *false* literal by definition, and one *true* literal since S was a satisfying truth assignment for F.

Suppose F' is NAE-satisfied with assignment S, we can then do a case distinction on the value of y. If $y = \textit{false}$ every clause in must F' contain a *true* literal not equal to y. Therefore, we can use the same truth assignment to satisfy F. If $y = \textit{true}$ then every clause in F' contains at least one *false* literal not equal to y. Therefore, if we invert the truth assignment by defining $x := \textit{true}$ if x was *false* in S, then every clause in F contains at least one *true* literal, thus F is satisfiable. ◀

▶ **Theorem 8.2.** *Let $d \geq 4$ be an integer. There is a linear parameter transformation from d-NAE-SAT to d-HYPERGRAPH 2-COLORABILITY. It follows that d-HYPERGRAPH 2-COLORABILITY does not have a kernel of size $O(n^{d-1-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$.*

Proof. Here we will use the result from Lemma 8.1 and give a linear parameter transformation from d-NAE-SAT to d-HYPERGRAPH 2-COLORABILITY. Suppose we are given a Boolean formula F in CNF-form for d-NAE-SAT, with variables $X = x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$. For hypergraph 2-colorability, we now create the set of vertices $V = \{y_1, \ldots, y_n\} \cup \{z_1, \ldots, z_n\}$. We create hyperedges e_i for $i \in [m]$ such that $y_i \in e_i$ if $x_i \in C_i$ and $z_i \in e_i$ if $\neg x_i \in C_i$. Then the following set of hyperedges is used for d-hypergraph 2-colorability.

$$E = \{e_i \mid 1 \leq i \leq m\} \cup \{\{y_i, z_i\} \mid 1 \leq i \leq n\}$$

It is easy to verify that every hyperedge in E has size at most d. Suppose F has a satisfying truth assignment T. Define color function c in the following way. Let $c(y_i) = \textit{Red}$, $c(z_i) = \textit{Blue}$ if $x_i = \textit{true}$ in T, and let $c(y_i) = \textit{Blue}$, $c(z_i) = \textit{Red}$ if $x_i = \textit{false}$ according to T. The hyperedges of type $\{y_i, z_i\} \in E$ are now 2-colored by definition. It is easy to verify that also every hyperedge e_i for $1 \leq i \leq m$ is 2-colored, since C_i contains at least one *true* and one *false* literal, resulting in one *Red* and one *Blue* vertex in e_i.

Suppose there is a proper 2-coloring $c \colon V \to \{\textit{Red}, \textit{Blue}\}$ of the hypergraph. Since for every $i \in [n]$ the hyperedge $\{y_i, z_i\}$ is present, the variables y_i and z_i must have a different color. Consider the following options

- y_i is *Red* and z_i is *Blue*. Then to satisfy F, we choose $x_i = \textit{true}$.
- z_i is *Red* and y_i is *Blue*. Then to satisfy F, we choose $x_i = \textit{false}$.

Since every edge e_i contains at least one *Red* vertex, this implies that every clause C_i for $i \in [m]$ contains at least one *true* literal using the assignment above. Also, since every hyperedge e_i contains at least one *Blue* vertex, every C_i must contain at least one *false* literal. Therefore F is NAE-satisfied with this assignment.

From the fact that linear-parameter transformations transfer lower bounds (Theorem 2.8) and Lemma 8.1 it now follows that d-HYPERGRAPH 2-COLORABILITY cannot have a kernel of size $\mathcal{O}(n^{d-1-\varepsilon})$ for any $\varepsilon > 0$, unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. ◀

8.2 Kernel

To prove validity of our kernel we use the following lemma due to Lovász [21]. This lemma was originally developed to prove bounds on the number of hyper edges in

critical 3-chromatic hypergraphs. We say a hypergraph G is critical 3-chromatic if G is not 2-colorable, but the removal of any edge from G yields a 2-colorable hypergraph. Lovász showed that for r-uniform critically 3-chromatic hypergraphs the number of edges is bounded by $\binom{n}{r-1} = \mathcal{O}(n^{r-1})$.

▶ **Lemma 8.3** ([21]). *Let H be an r-uniform hypergraph with hyperedges $E_1, \ldots, E_m$. Let $\alpha_1, \ldots, \alpha_m$ be real numbers such that for every $(r-1)$-element subset A of $V(H)$,*

$$\sum_{E_i \supset A} \alpha_i = 0.$$

Then for every partition $\{V_1, V_2\}$ of $V(H)$ the following holds:

$$\sum_{E_i \subseteq V_1} \alpha_i = (-1)^r \sum_{E_i \subseteq V_2} \alpha_i.$$

◀

▶ **Theorem 8.4.** *d-*HYPERGRAPH 2-COLORABILITY *parameterized by the number of vertices n has a kernel with $2 \cdot n^{d-1}$ hyperedges that can be encoded in $\mathcal{O}(n^{d-1} \cdot d \cdot \log n)$ bits.*

Proof. Suppose we are given a hypergraph with vertex set V and hyperedges E, where each hyperedge contains at most d vertices. We show how to reduce the number of hyperedges without changing the 2-colorability status. Let $E_r \subseteq E$ denote the set of edges in E that contain exactly r vertices. For each E_r we construct a set $E'_r \subseteq E_r$ of *representative hyperedges*. Enumerate the edges in E_r as $e^r_1, \ldots, e^r_k$. We construct a $(0,1)$-matrix M_r with $N := \binom{n}{r-1}$ rows and k columns. Consider all possible subsets $A_1, \ldots, A_N$ of size $r-1$ of the set of vertices V. Define the elements $m_{i,j}$ for $i \in N$ and $j \in k$ of M_r as follows.

$$m_{i,j} := \begin{cases} 1 & \text{if } A_i \subseteq e^r_j; \\ 0 & \text{otherwise.} \end{cases}$$

Using Gaussian elimination, compute a base B of the columns of this matrix, which is a subset of the columns that span the column space of M_r. Let E'_r contain edge e^r_i if the i'th column of M_r is contained in B, and define $E' := \bigcup_{r \in [d]} E'_r$, which forms the kernel. We will now prove validity of the obtained kernel.

▷ **Lemma 8.5.** *(V, E) has a proper 2-coloring $\Leftrightarrow$ (V, E') has a proper 2-coloring.*

Proof. ($\Rightarrow$) Clearly, if (V, E) has a proper 2-coloring, then the same coloring is proper for the subhypergraph (V, E') since $E' \subseteq E$.

($\Leftarrow$) Now suppose (V, E') has a proper 2-coloring. We say a hyperedge is *monochromatic* with respect to some coloring, if all its vertices receive the same color. We will show that for each $r \in [d]$, no hyperedge of E_r is monochromatic under this coloring. All hyperedges contained in E'_r are 2-colored by definition. Suppose there exists $r \in [d]$,

such that E_r contains a monochromatic hyperedge. Let $E_r = e_1^r, \ldots, e_k^r$ and let e_{i^*} be a hyperedge in E_r whose vertices all receive the same color.

By reordering the matrix M_r, we may assume that the basis B of M_r contains the first ℓ columns, thus $i^* > \ell$. Let $\mathbf{m}_i$ denote the i'th column of M_r. Since $\mathbf{m}_{i^*}$ is not contained in the basis, there exist coefficients $\alpha_1, \ldots, \alpha_\ell$ such that

$$\sum_{i=1}^{\ell} \alpha_i \cdot \mathbf{m}_i = \mathbf{m}_{i^*}.$$

For $i \in [k]$, define:

$$\beta_i := \begin{cases} \alpha_i & \text{if } i \leq \ell; \\ -1 & \text{if } i = i^*; \\ 0 & \text{otherwise.} \end{cases}$$

From this definition of β it follows that

$$\sum_{i=1}^{k} \beta_i \cdot \mathbf{m}_i = \sum_{i=1}^{\ell} \alpha_i \cdot \mathbf{m}_i - \mathbf{m}_{i^*} = \mathbf{0}.$$

Let A_j be any size $(r-1)$-subset of V. Since $m_{i,j} = 1$ exactly when $e_i \supseteq A_j$,

$$\sum_{e_i \supset A_j} \beta_i = \sum_{i=1}^{k} \beta_i m_{i,j} = 0.$$

By Lemma 8.3 we obtain that for any partitioning $V_1 \cup V_2$ of the vertices in V,

$$\sum_{e_i \subseteq V_1} \beta_i = (-1)^r \sum_{e_i \subseteq V_2} \beta_i. \tag{1}$$

Consider however the partitioning (V_1, V_2) given by the 2-coloring of the vertices. Then every edge $e_i \in E'_r$ contains at least one vertex of each color and is thereby not fully contained in V_1 or V_2. As such, these edges contribute 0 to both sides of the equation. The edge e_{i^*} is the only remaining edge with a non-zero coefficient and by assumption, it is contained entirely within one color class. Without loss of generality, let $e_{i^*} \subseteq V_1$. But then $\sum_{e_i \subseteq V_1} \beta_i = -1$ while $(-1)^r \sum_{e_i \subseteq V_2} \beta_i = 0$, which contradicts Formula (1). ◁

▷ **Lemma 8.6.** *The obtained kernel contains at most* $2 \cdot n^{d-1}$ *hyperedges.*

Proof. Consider matrix M_r for $r \in [d]$. Its rank is bounded by the minimum number of its rows and columns, which is at most $\binom{n}{r-1} \leq n^{r-1}$. As such, we get that $|E'_r| \leq \text{rank}(M_r) \leq n^{r-1}$. Note that $d \leq n$, such that

$$|E'| \leq \sum_{r=1}^{d} n^{r-1} = n^{d-1} + \sum_{r=1}^{d-1} \leq 2 \cdot n^{d-1}.$$

So E' contains at most $2n^{d-1}$ hyperedges. ◁

It follows from Lemma 8.5 and 8.6 that we have presented a kernel consisting of $2 \cdot n^{d-1}$ hyperedges and n vertices. Thereby, we can store the representation of a vertex in $\mathcal{O}(\log n)$ bits. Since every hyperedge consists of at most d vertices, we can store this kernel in $\mathcal{O}(d \cdot \log n \cdot n^{d-1})$ bits. ◀

As we have seen in Theorem 8.2, we can give a linear parameter transformation from d-NAE-SAT to d-HYPERGRAPH 2-COLORABILITY. Suppose we are given an instance of d-NAE-SAT, we can now obtain a generalized kernel for this problem by transforming it to an instance of d-HYPERGRAPH 2-COLORABILITY and then applying the kernel described in Theorem 8.4. This results in an instance of at most $\mathcal{O}(n^{d-1})$ hyperedges.

9 Conclusion

We have shown that several well-known graph problems do not have a generalized kernel of size $\mathcal{O}(n^{2-\varepsilon})$, unless NP $\subseteq$ coNP/poly. For example, we showed that 4-coloring does not have a generalized kernel of sub-quadratic size. However, it is left as an open question whether this also holds for the 3-coloring problem, that is also known to be NP-hard. We also showed that a kernel of sub-quadratic size for one of the problems HAMILTONIAN CYCLE, (CONNECTED) DOMINATING SET, MAX LEAF SPANNING TREE, NONBLOCKER, and FEEDBACK ARC SET would imply that NP $\subseteq$ coNP/poly.

Most graph problems we considered ask a question about the vertices of the graph, such as 4-coloring which asks if there exists a proper coloring of the vertices. For edge-based problems, such as the maximum cut problem that asks for a cut in the graph of size k, it seems harder to find a cross-composition using the same strategy as we used in this thesis. The approach to select one instance and somehow remove the effect of all other instances does not seem to work out. Only for FEEDBACK ARC SET, we have shown a lower bound using a linear parameter transformation.

Furthermore, we showed a lower bound for d-NAE-SAT and d-HYPERGRAPH 2-COLORABILITY and a nontrivial sparsification to match the lower bound, using a lemma by Lovász. It is an interesting open question whether there also exist problems defined on general graphs that allow a non-trivial sparsification, since so far we have only given examples of problems that do *not* have such a sparsification.

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